Midterm Exam - MAS 5145 Fall 2011 **NAME:** ________________________________

To receive credit you MUST SHOW ALL YOUR WORK.

1. (18 pts) Let $L : V \to W$ be a linear map between vector spaces.

(a) (8 pts) Define $Ker(L)$ and show it is a subspace of V;

(b) (10 pts) Define the nullity of L, the rank of L, state the rank-nullity theorem and prove it (in one line).

2. (14 pts) (a) (8 pts) Find a 2×2 matrix A which has eigenvector $\mathbf{v}_1 = (2, 1)^T$ with eigenvalue $\lambda_1 = 3$ and eigenvector $\mathbf{v}_2 = (3, 2)^T$ with eigenvalue $\lambda_2 = 1$.

(b) (6 pts) Compute A^n, where A is the matrix in part (a).

3. (16 pts) It is determined that a predator-prey model in a certain ecosystem evolves according with the equations

$$\begin{cases} x_1(n+1) = & 0.3x_1(n) + 0.6x_2(n) \\ x_2(n+1) = & -0.1x_1(n) + x_2(n) \end{cases},$$

where n denotes the number of years since an initial moment $n = 0$, and $x_1(n), x_2(n)$ denote the two populations in year n.

(a) (2 pts) Between $x_1(n), x_2(n)$, which one must be the predator population, which one must be the prey population?

(b) (10 pts) Assuming known the initial data $x_1(0), x_2(0)$, find the formulae for $x_1(n)$, $x_2(n)$.

(c) (4 pts) What happens with the two populations in the long run? Justify your answer. Do the initial conditions matter?

4. (16 pts) Let A, B be $n \times n$ real matrices.

(a) (10 pts) Show that if A is similar to B, then A and B have the same characteristic polynomial and, hence, the same eigenvalues.

(b) (6 pts) Do you think the converse is true? That is, if A and B have the same characteristic is it true that A and B are similar? Justify your answer.

Note: *Similar matrices* are called in Sadun's text *conjugate matrices.* We'll more often use *similar matrices*, as it is the wider accepted terminology.

5. (18 pts) Let $L : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$ be the linear operator defined by $L(p(t)) = t(p(t+1) - p(t))$.

(a) (4 pts) Find the matrix $[L]_{\mathcal{E}}^{\mathcal{E}}$, where $\mathcal{E} = \{1, t, t^2\}$.

(b) (2 pts) Determine $rank(L)$.

(c) (6 pts) Is L diagonalizable? If yes, find a basis of $\mathbb{R}_2[t]$ that diagonalizes L, if not, explain why such basis does not exist.

(d) (6 pts) More generally, is the operator $L(p(t)) = t(p(t+1) - p(t))$ diagonalizable when $L : \mathbb{R}_n[t] \to \mathbb{R}_n[t]$? What about the case $L : \mathbb{R}[t] \to \mathbb{R}[t]$?

Hint: How does $[L]_{\mathcal{E}}^{\mathcal{E}}$ look like in this cases? Just give a brief justification based on this.

Choose one and indicate your choice:

6. (14 pts) Let V be a real vector space.

(a) Define the dual V' and prove that if V is finite dimensional, V and V' are isomorphic vector spaces.

(b) Show also that there is a natural injective linear map from V to $V'' = (V')'$ and this map is an isomorphism when V is finite dimensional.

6'. (14 pts) State the Fundamental Isomorphism Theorem and use it to prove:
The 2nd Isomorphism Theorem: Suppose V is a vector space and that S and T are subspaces in V. Then $S/(S \cap T)$ and $(S+T)/T$ are naturally isomorphic vector spaces.

In the case when S and T are finite dimensional, deduce from the 2nd Iso Theorem another proof of the relation

$$dim(S+T) = dim(S) + dim(T) - dim(S \cap T) \ .$$

Note: You may use without proof that $S \cap T$ and $S+T$ are subspaces, as well as the dimension formula for a quotient vector space.

Choose one and indicate your choice:

7. (14 pts) In the attached copy, you have Peter Lax's argument showing the existence and uniqueness of the discrete Dirichlet problem on a bounded domain for the Laplace operator.
Can the argument be adjusted to work if we replace the usual Laplace operator $\Delta u = u_{xx} + u_{yy}$, with a weighted Laplace operator $\Box u = a u_{xx} + u_{yy}$, where $a > 0$? If yes, explain what changes are needed, if no, explain what part of the argument breaks down. To be concrete, you could take $a = 3$.

7'. (14 pts) Show that for any matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$, there exist a polynomial $q \in \mathbb{R}[t]$, so that $q(A) = 0$, that is, if the variable t in the polynomial is replaced by the matrix A, then $q(A)$ is the zero matrix.

Note: If you use a theorem, you should prove it. Here is a hint to do this without any special theorem: consider the set of matrices $\{I, A, A^2, A^3, ...\}$ in the space $\mathcal{M}_{n,n}(\mathbb{R})$; what is $dim(\mathcal{M}_{n,n}(\mathbb{R}))$?