NAME: Answer Key Panther ID: ____________

Exam 2 - MAC 2311 Fall 2013

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) The function $h(x)$ is given by $h(x) = \frac{x}{f(x)}$. It is known that $f(3) = -1$ and $f'(3) = 2$. Compute

(a) (3pts) $h(3)$

$$h(3) = \frac{3}{f(3)} = \frac{3}{(-1)} = \boxed{-3}$$

(b) (7pts) $h'(3)$

$$h'(x) = \left(\frac{x}{f(x)}\right)' = \frac{1\cdot f(x) - x f'(x)}{(f(x))^2} \quad \text{quotient rule}$$

$$h'(3) = \frac{f(3) - 3\cdot f'(3)}{(f(3))^2} = \frac{-1 - 3\cdot 2}{(-1)^2} = \boxed{-7}$$

2. (10 pts) Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

As $y = \frac{1}{x} = x^{-1}$, $y' = -x^{-2} = -\frac{1}{x^2}$ and

$$y'' = 2x^{-3} = \frac{2}{x^3}$$

Plugging these in the left hand-side of the equation, we get

$$x^3y'' + x^2y' - xy = x^3\cdot\frac{2}{x^3} + x^2\cdot\left(-\frac{1}{x^2}\right) - x\cdot\frac{1}{x} = 2 - 1 - 1 = 0$$

3. (35 pts) Find the derivative of each of the following functions. Simplify your answer when possible (7 pts each):

(a) $y = 4x^3 - 2x\sqrt{x} + \frac{\pi^2}{2}$

$$y = 4x^3 - 2x^{\frac{3}{2}} + \frac{\pi^2}{2}$$

$$y' = 12x^2 - 2 \cdot \frac{3}{2} x^{\frac{1}{2}} + 0$$

$$\boxed{y' = 12x^2 - 3\sqrt{x}}$$

(b) $y = \ln(\cos(3x))$

$$y' = (\ln(\cos(3x)))' \overset{\text{chain Rule}}{=}$$

$$= \frac{1}{\cos(3x)} \cdot (-\sin(3x)) \cdot 3$$

so $\boxed{y' = -3\tan(3x)}$

(c) $y = e^{-3x} \sin x$

using product rule,

$$y' = (e^{-3x})' \sin x + e^{-3x} \cdot (\sin x)'$$

$$y' = -3e^{-3x} \sin x + e^{-3x} \cos x$$

$$\boxed{y' = e^{-3x}(-3\sin x + \cos x)}$$

(d) $y = \arcsin\left(\frac{1}{x}\right)$ $= \arcsin(x^{-1})$

$$y' = (\arcsin(x^{-1}))' \overset{\text{Chain Rule}}{=}$$

$$= \frac{1}{\sqrt{1-(x^{-1})^2}} \cdot (-x^{-2})$$

$$y' = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(-\frac{1}{x^2}\right)$$

$$y' = \frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right)$$

$$\boxed{y' = -\frac{1}{|x|\sqrt{x^2-1}}}$$

<u>Note</u>: It can be shown (try this as an exercise) that

$$\arcsin\left(\frac{1}{x}\right) = \operatorname{arccsc} x$$

Thus the formula you got for y' above is not surprising

(e) $y = x^{2^x}$

This requires log. differentiation.

$$\ln y = \ln\left(x^{2^x}\right) = 2^x \cdot \ln x$$

so $\frac{d}{dx}(\ln y) = \frac{d}{dx}(2^x \cdot \ln x)$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2^x \cdot \ln 2 \cdot \ln x + 2^x \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y \cdot 2^x \cdot \left(\ln 2 \cdot \ln x + \frac{1}{x}\right)$$

so $\boxed{\frac{dy}{dx} = x^{2^x} \cdot 2^x \cdot \left(\ln 2 \cdot \ln x + \frac{1}{x}\right)}$

4. (15 pts) (a) (8 pts) Use implicit differentiation to find dy/dx for the (rotated) ellipse $x^2 - xy + y^2 = 3$.

(b) (7 pts) Find all points on the ellipse $x^2 - xy + y^2 = 3$ where the tangent line is vertical.

(a) Apply $\frac{d}{dx}$ to both sides of $x^2 - xy + y^2 = 3$. We get

$$2x - (1 \cdot y + x \cdot y') + 2y \cdot y' = 0$$

$$2x - y = xy' - 2y \cdot y'$$

so $\boxed{y' = \dfrac{2x - y}{x - 2y}}$

(b) For the tangent line to be vertical, its slope should be undefined, thus we must have $x - 2y = 0$. Substituting $x = 2y$ in $x^2 - xy + y^2 = 3$, we get $4y^2 - 2y^2 + y^2 = 3 \Rightarrow 3y^2 = 3 \Rightarrow y = \pm 1$.

Thus we have two points: $(2, 1)$ and $(-2, -1)$.

5. (10 pts) Find the equation of the tangent line to the hyperbola $x = \sec t$, $y = 2\tan t$ when $t = \pi/4$.

Solution 1: Using parametric curves

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sec^2 t}{\sec t \cdot \tan t} = 2\frac{\sec t}{\tan t}$$

$$m_{tan} = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = 2\frac{\sec(\frac{\pi}{4})}{\tan(\frac{\pi}{4})} = 2\sqrt{2}$$

Point $(\sec\frac{\pi}{4}, 2\tan\frac{\pi}{4}) = (\sqrt{2}, 2)$

$\Rightarrow$ Tangent line $\boxed{y - 2 = 2\sqrt{2}(x - \sqrt{2})}$

Solution 2: Eliminating the parameter and using implicit differentiation

Use the identity $\sec^2 t - \tan^2 t = 1$ to eliminate parameter t:

$$x^2 - \left(\frac{y}{2}\right)^2 = 1 \quad \text{or} \quad x^2 - \frac{y^2}{4} = 1$$

Next compute $\frac{dy}{dx}$ by implicit differentiation: $2x - \frac{2y}{4} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{4x}{y}$. As the point is $(\sqrt{2}, 2)$, the slope $m = \frac{4 \cdot \sqrt{2}}{2} = 2\sqrt{2}$

6. (8 pts) Find a formula for the **second** derivative of a composite function $f(g(x))$.

$$(f(g(x)))' = f'(g(x))\cdot g'(x) \quad \text{Chain Rule}$$

$$(f(g(x)))'' \underset{\substack{\text{prod.}\\\text{rule}}}{=} (f'(g(x)))'\cdot g'(x) + f'(g(x))\cdot(g'(x))' =$$

$$= f''(g(x))\cdot g'(x)\cdot g'(x) + f'(g(x))\cdot g''(x) = \underline{f''(g(x))\cdot(g'(x))^2 + f'(g(x))\cdot g''(x)}$$

7. (12 pts) The volume of a sphere of radius r is given by $V = \frac{4\pi r^3}{3}$. A spherical balloon is inflated so that its volume is increasing at a constant rate of π ft^3/min.

(a) (8 pts) How fast is the radius of the balloon increasing when the diameter of the balloon is 2 ft?

(b) (4 pts) Is the radius of the balloon increasing at the same rate at all times? Justify your answer.

(a) It is given that $\frac{dV}{dt} = \pi$. We are asked $\frac{dr}{dt} = ?$ when $r = \frac{2}{2} = 1$ ft

Differentiating w.r.t. "t" $V = \frac{4\pi}{3} r^3$, we get

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{3}}\cdot \cancel{3} r^2 \frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}} \; (*)$$

when $r = 1$ ft $\Rightarrow \frac{dr}{dt} = \frac{\pi}{4\pi\cdot 1} = \frac{1}{4}$ ft/min.

(b) No, r does not increase at a constant rate.
From (*) above one can see that the rate of growth of r gets smaller as r increases.

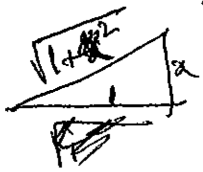
8. (10 pts) Find with proof a formula for $(\arctan x)'$.

Let $y = \arctan x \Rightarrow \tan y = x$

Use implicit differentiation now

$$\sec^2 y \cdot y' = 1 \Rightarrow y' = \frac{1}{\sec^2 y} = \cos^2 y$$

But if $\tan y = x = \frac{x}{1}$



$$\Rightarrow \cos y = \frac{1}{\sqrt{1+x^2}}$$

Thus $\boxed{(\arctan x)' = \frac{1}{1+x^2}}$.