NAME: Solution Key Panther ID: ____________

Exam 3 - MAC 2311 Fall 2013

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (6 pts) Fill in the most appropriate words:

(a) If x_0 is a critical point for the function $f(x)$, then $f'(x_0)$ is zero or undefined

(b) If $f''(x) < 0$, for all $x \in (a, b)$, then on the interval (a, b) the function f is concave down.

(c) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a relative minimum (local) for the function $f(x)$.

2. (16 pts) Compute each of the following limits:

(a) $\lim_{x\to+\infty} x\tan\left(\frac{5}{x}\right) =$ $\infty \cdot 0$

$= \lim_{x\to+\infty} \frac{\tan\left(\frac{5}{x}\right)}{\frac{1}{x}}$ $\frac{0}{0}$ so now apply l'Hopital

$= \lim_{x\to+\infty} \frac{\sec^2\left(\frac{5}{x}\right)\cdot\cancel{\left(-\frac{5}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}} =$

$= 5\sec^2(0) = \boxed{5}$

Alternative way from here:
substitute $y = \frac{1}{x}$
(so if $x \to +\infty$, $y \to 0^+$)

$\lim_{y\to 0^+} \frac{\tan(5y)}{y} = 5$

from previous knowledge (or from l'H)

(b) $\lim_{x\to 0} (e^x + 2x)^{1/x}$ 1^∞

$= \lim_{x\to 0} e^{\ln\left[(e^x+2x)^{\frac{1}{x}}\right]} = \lim_{x\to 0} e^{\left[\frac{\ln(e^x+2x)}{x}\right]}$

$= e^{\lim_{x\to 0}\frac{\ln(e^x+2x)}{x}} = \boxed{e^3}$

$\lim_{x\to 0} \frac{\ln(e^x+2x)}{x}$ $\frac{0}{0}$ $\overset{l'H}{=}$

$= \lim_{x\to 0} \frac{\frac{1}{e^x+2x}\cdot(e^x+2)}{1} =$

$= \frac{e^0+2}{e^0+2\cdot 0} = \frac{3}{1} = 3$

3. (12 pts) (a) (8 pts) Find the local linear approximation of the function $f(x) = \tan x$ at $x = \pi/4$.

Tangent line at point $(x_0, f(x_0))$ has equation $y - f(x_0) = f'(x_0)(x - x_0)$, so the local linear approximation formula is $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

In your case, $f(x) = \tan x$ and $x_0 = \frac{\pi}{4}$

$f(x_0) = \tan\left(\frac{\pi}{4}\right) = 1$ $\quad f'(x) = (\tan x)' = \sec^2 x$ so $f'(x_0) = \sec^2\left(\frac{\pi}{4}\right) = 2$

Thus, $\boxed{\tan x \approx 1 + 2\left(x - \frac{\pi}{4}\right)}$ where the approximation is good when x is close to $\frac{\pi}{4}$.

(b) (4 pts) Use part (a) to approximate $\tan 43°$ without using a calculator. (It's OK if your answer contains π.)

$43° = 43 \cdot \frac{\pi}{180}$ radians, so using (a)

$$\tan(43°) = \tan\left(\frac{43\pi}{180}\right) \approx \boxed{1 + 2\left(\frac{43\pi}{180} - \frac{\pi}{4}\right)} \overset{\text{or}}{=} 1 + 2\left(\frac{43\pi}{180} - \frac{45\pi}{180}\right)$$

$$= 1 - 2 \cdot \frac{2\pi}{180} = \boxed{1 - \frac{\pi}{45}}$$

4. (20 pts) Find the indicated antiderivatives:

(a) $\int \left(3\sqrt{x} - \frac{e^x}{2} + \frac{1}{1+x^2}\right) dx = \int \left(3x^{\frac{1}{2}} - \frac{1}{2}e^x + \frac{1}{1+x^2}\right) dx$

$$= \cancel{3} \cdot \frac{2}{\cancel{3}} x^{\frac{3}{2}} - \frac{1}{2}e^x + \arctan x + C = \underline{2x\sqrt{x} - \frac{1}{2}e^x + \arctan x + C}$$

(b) $\int \frac{x^2}{2x^3 + 1}\, dx =$

subst. $w = 2x^3 + 1$

$dw = 6x^2\, dx$

$\frac{1}{6}dw = x^2\, dx$

$$\overset{S}{=} \int \frac{\frac{1}{6}dw}{w} = \frac{1}{6}\int \frac{1}{w}\, dw$$

$$= \boxed{\frac{1}{6}\ln(2x^3 + 1) + C}$$

(c) $\int \sin^4(5t)\cos(5t)\, dt =$

sub. $w = \sin(5t)$

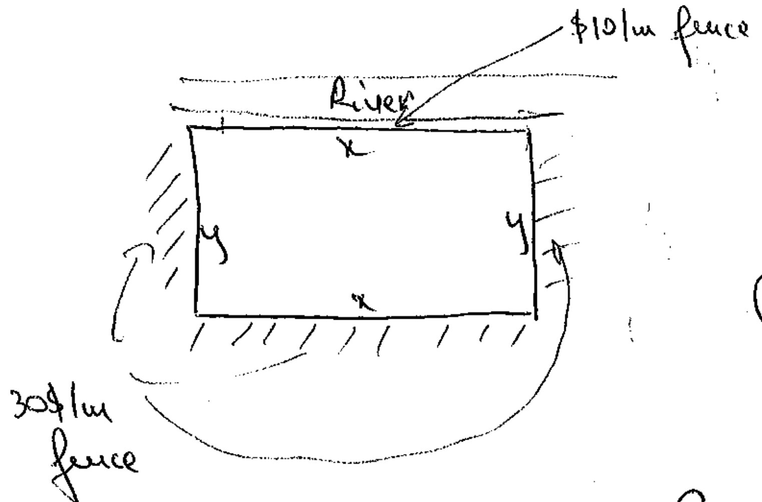
$dw = 5\cos(5t)\, dt$

$\frac{1}{5}dw = \cos(5t)\, dt$

$$= \int w^4 \frac{1}{5}\, dw = \frac{1}{5} \cdot \frac{1}{5} w^5 + C$$

$$= \boxed{\frac{1}{25}\sin^5(5t) + C}$$

5. (14 pts) Suppose you are allowed to choose a rectangular plot of land along a (straight) river. The rectangular plot is to have an area of 3000 square meters. You are required to fence in your land using two kinds of fencing. Three of the four sides will use heavy-duty fencing selling for \$30 per meter while the remaining side (along the river) will use standard fencing selling for \$10 per meter. How should you choose the dimensions of your plot of land in order to minimize the cost of fencing? (It's OK if your result contains square-roots.)



$A = x \cdot y = 3000$

$\Rightarrow y = \dfrac{3000}{x}$

Cost of fence = C

$C = 30 \cdot x + 30 \cdot (2y) + 10 \cdot x$

$C = 30x + 60y + 10x = 40x + 60y$

Use $y = \dfrac{3000}{x}$ and substitute.

$C(x) = 40x + 60 \cdot \dfrac{3000}{x} = 40x + \dfrac{180,000}{x}$

Domain of $C(x)$ $\quad x \in (0, +\infty)$

We want to find the absolute minimum of $C(x)$.

$C'(x) = 40 - \dfrac{180,000}{x^2}$ $\qquad C'(x) = 0 \Leftrightarrow 40 = \dfrac{180,000}{x^2}$

so $x^2 = \dfrac{\cancel{180,000}^{4500}}{\cancel{40}} = 4500$

$x = \sqrt{4500} = 30\sqrt{5} \leftarrow$ the only critical point in the domain

Since $\lim_{x \to 0^+} C(x) = +\infty$ and $\lim_{x \to +\infty} C(x) = +\infty$, the critical point must be an absolute minimum.

Thus, the dimensions of the plot that minimize the cost are

$x = 30\sqrt{5}$ m, $\quad y = \dfrac{3000}{30\sqrt{5}} = \dfrac{100}{\sqrt{5}} = \dfrac{\cancel{100}^{20}\sqrt{5}}{\cancel{5}} = 20\sqrt{5}$ m,

with the side x chosen along the river.

6. (10 pts) Verify that the hypothesis of the Mean-Value Theorem are satisfied for the function $f(x) = \sqrt{9-x}$ on the interval $[0, 9]$, and find the value(s) of c in the interval that satisfy the conclusion of the Theorem.

$f(x)$ is defined & continuous for $x \le 9$, thus $f(x)$ is continuous for $x \in [0,9]$.

$f'(x) = \left((9-x)^{\frac{1}{2}}\right)' = \frac{1}{2}(9-x)^{-\frac{1}{2}} \cdot (-1) = -\frac{1}{2\sqrt{9-x}}$

so $f'(x)$ is defined for $x < 9$, thus $f(x)$ is differentiable for $x \in (0,9)$

MVT says that there must be a point $c \in (0,9)$ s.t. $f'(c) = \frac{f(9)-f(0)}{9-0}$

This means $-\frac{1}{2\sqrt{9-c}} = \frac{0-3}{9} \iff -\frac{1}{2\sqrt{9-c}} = -\frac{1}{3} \iff 2\sqrt{9-c} = 3 \iff \sqrt{9-c} = \frac{3}{2}$

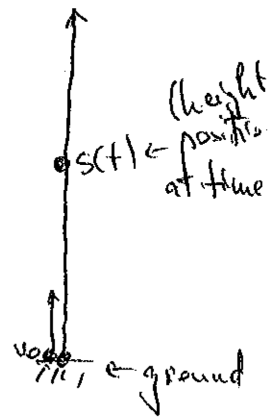
Thus $9-c = \frac{9}{4} \Rightarrow \boxed{c = 9 - \frac{9}{4} = \frac{27}{4}}$. Obviously, $\frac{27}{4} \in (0,9)$, so $c = \frac{27}{4}$ satisfies conclusion of MV

7. (12 pts) A baseball is thrown straight upward from ground level with an initial velocity of 96 ft/s.

(a) (8 pts) Use integration to find the formulas for the velocity $v(t)$ and the position $s(t)$ of the baseball at time t. Assume gravitational acceleration $g = -32$ ft/s^2.

(b) (4 pts) When does the ball reach the maximum height?

(a) Given: $a = -32$ (constant acceleration), initial velocity $v(0) = 96$, initial position $s(0) = 0$



$v(t) = \int a\, dt = \int -32\, dt = -32t + c$

$96 = v(0) = -32 \cdot 0 + c$ so $c = 96$

Thus $\boxed{v(t) = -32t + 96}$

$s(t) = \int v(t)\, dt = \int (-32t + 96)\, dt = -32\frac{t^2}{2} + 96t + c$

$0 = s(0) = c$

so $\boxed{s(t) = -16t^2 + 96t}$

(b) Max height is achieved when $s'(t) = v(t) = 0$

Thus $-32t + 96 = 0$, so $\boxed{t = \frac{96}{32} = 3 \text{ seconds}}$

8. (20 pts) The steps of this problem should lead you to a complete graph of the function $f(x) = \frac{\ln x}{x}$. Where indicated, work should be shown below, or on a separate sheet of paper.

(a) (2 pts) The domain of this function is $x \in (0, +\infty)$

(b) (3 pts) The derivative is $f'(x) = \boxed{\frac{1-\ln x}{x^2}}$. Show work.

$$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - 1\cdot \ln x}{x^2}$$

(c) (3 pts) Critical point(s) of f: $\boxed{x = e}$. Show work.

$\to$ the only critical point

$$f'(x) = 0 \iff 1 - \ln x = 0 \iff \ln x = 1 \iff \boxed{x = e}$$

(d) (3 pts) Do a sign chart for f' and mark the intervals where f is increasing, respectively decreasing.

x	0		e		$+\infty$
$f'(x)$		$+ + + +$	0	$- - - - -$	
$f(x)$		$\nearrow$	$\frac{\ln e}{e} = \frac{1}{e}$	$\searrow$	

So $f(x)$ increasing on $(0, e)$

$f(x)$ decreasing on $(e, +\infty)$

Also follows from here that $x = e$ is a local & global maximum

(e) (4 pts) End behavior: $\lim_{x\to 0^+} \frac{\ln x}{x} = \frac{-\infty}{0^+} = -\infty$ $\qquad \lim_{x\to+\infty} \frac{\ln x}{x} = \frac{+\infty}{+\infty}$ Show work below.

graph has vertical asymptote at $x = 0$

(i.e. y-axis is a vertical asymptote)

l'H "

$$\lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$$

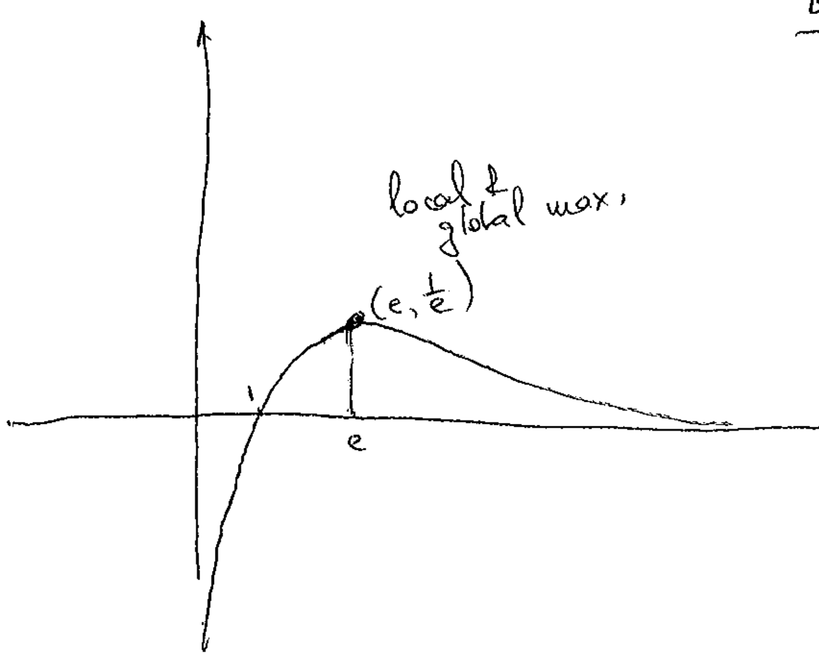
$\to$ Thus $\boxed{\lim_{x\to+\infty} \frac{\ln x}{x} = 0}$

$y = 0$ is a horiz. asympt

(i.e. x-axis is horiz. asymptote)

(f) (5 pts) Using all the previous steps, sketch the graph of $f(x) = \frac{\ln x}{x}$. Label on the graph the coordinates of critical point(s) and also specify the type of the critical point.

Bonus 2pts: I did not ask you to do the analysis of the second derivative. Without computing the second derivative, how many inflection points do you expect?



Bonus: There must be a change in concavity somewhere in the interval $(e, +\infty)$

so there is one inflection point in this interval.

Its exact location can be found by computing the second derivative, etc.