Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus I Spring 2013

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (24 pts) In each case, find dy/dx. Simplify when possible.

(a) $y = 2x^3 + \frac{1}{3x^3} - \pi^2 = 2x^3 + \frac{1}{3}x^{-3} - \pi^2$

$y' = \left(2x^3 + \frac{1}{3}x^{-3} - \pi^2\right)'$

$y' = 6x^2 - x^{-4} = \boxed{6x^2 - \frac{1}{x^4}}$

(b) $y = x^2 \arctan x$

product rule

$y' = (x^2 \arctan x)' = 2x \arctan x + x^2 \cdot \frac{1}{1+x^2}$

$\boxed{y' = 2x \arctan x + \frac{x^2}{1+x^2}}$

(c) $y = x^{\sin x}$

logarithmic differentiation

$\ln y = \ln(x^{\sin x})$

$\ln y = \sin x \cdot \ln x$ | Differentiate both sides

$(\ln y)' = (\sin x \cdot \ln x)'$

$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$

$y' = x^{\sin x}\left[\cos x \ln x + \frac{\sin x}{x}\right]$

(d) $y = \ln(\sec x + \tan x)$

$y' = \left(\ln(\sec x + \tan x)\right)'$ Chain Rule

$y' = \frac{1}{\sec x + \tan x} \cdot (\sec x + \tan x)'$

$y' = \frac{1}{\sec x + \tan x}\left(\sec x \tan x + \sec^2 x\right)$

$y' = \frac{1}{\cancel{\sec x + \tan x}} \cdot \sec x\left(\cancel{\tan x + \sec x}\right)$

$\boxed{y' = \sec x}$

2. (12 pts) (a) Write the definition with limit of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

For each of the following, fill in the blanks with the appropriate words or expressions:

(b) A function $f(x)$ is continuous at a point $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

(c) The *average* rate of change of a function $f(x)$ on an interval $x \in [a, b]$ is $\frac{f(b)-f(a)}{b-a}$.

(d) A point on a graph of a function where concavity changes is called inflection point.

(e) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on the interval (a, b).

3. (20 pts) Find the following limits. Be sure to specify if a limit is infinite or does not exists.

(a)(6 pts) $\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}$

$$\overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{x(\sin x)\cdot(1+\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0} \frac{x \sin x (1+\cos x)}{1-\cos^2 x} = \lim_{x\to 0} \frac{x \cancel{\sin x} (1+\cos x)}{\sin^{\cancel{2}} x} =$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x} \cdot (1+\cos x)\right) = 1.2 = 2$$

(With l'Hopital also OK, but longer)

(b)(6 pts) $\lim_{x\to 3} \frac{3x - x^2}{|x - 3|}$ → should do the two one sided limits

$$\lim_{x\to 3^+} \frac{3x-x^2}{|x-3|} = \lim_{x\to 3^+} \frac{x\cancel{(3-x)}(-1)}{\cancel{(x-3)}} = -3$$

$$\lim_{x\to 3^-} \frac{3x-x^2}{|x-3|} = \lim_{x\to 3^-} \frac{x\cancel{(3-x)}}{\cancel{3-x}} = 3$$

Thus $\boxed{\lim_{x\to 3} \frac{3x-x^2}{|x-3|} \quad \text{D.N.E.}}$

(c) (8 pts) $\lim_{x\to\infty} \left(1 - \frac{2}{x}\right)^{3x}$

1^∞ so apply the $A = e^{\ln A}$ trick

$$\lim_{x\to+\infty} e^{\ln\left(\left(1-\frac{2}{x}\right)^{3x}\right)} = \lim_{x\to+\infty} e^{3x\ln\left(1-\frac{2}{x}\right)} = e^{\lim_{x\to+\infty} 3x\ln\left(1-\frac{2}{x}\right)}$$

$$\lim_{x\to+\infty} 3x \ln\left(1-\frac{2}{x}\right) \underset{\infty\cdot 0}{=} \lim_{x\to+\infty} \frac{3\ln\left(1-\frac{2}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{l'H}{=}}$$

$$= \lim_{x\to+\infty} \frac{3\cdot \frac{1}{1-\frac{2}{x}} \cdot \frac{\cancel{2}\,2}{\cancel{x^2}}}{-\frac{1}{\cancel{x^2}}\,(-1)} = -6$$

Thus $\boxed{\lim_{x\to+\infty} \left(1-\frac{2}{x}\right)^{3x} = e^{-6}}$

4. (20 pts) Find each indicated antiderivative:

(a)(6 pts) $\int \left(3-\sec^2 x+\frac{2}{x}\right) dx$

$= 3x - \tan x + 2\ln|x| + c$

(b)(6 pts) $\int \frac{x}{\sqrt{x^2+1}}\, dx = \int \frac{\frac{1}{2}dw}{\sqrt{w}} = \frac{1}{2}\int w^{-\frac{1}{2}}dw = \frac{1}{2}\cdot 2\cdot w^{\frac{1}{2}} + c$

substitution

$w = x^2+1$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$= w^{\frac{1}{2}} + c = \boxed{\sqrt{x^2+1} + c}$

(c)(8 pts) $\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$

$= \int (1-\sin^2 x)\cos x\, dx =$

sub. $u = \sin x$

$du = \cos x\, dx$

$= \int (1-u^2)du = u - \frac{u^3}{3} + c =$

$= \boxed{\sin x - \frac{\sin^3 x}{3} + c}$

5. (10 pts + 8 bonus) Find the equation of the tangent line to the ellipse $x = \sqrt{3}\cos t$, $y = 2\sin t$ when $t = \pi/6$. (You can do this problem either using parametric equations, or by eliminating the parameter and then using implicit differentiation. If you correctly do the problem both ways, you receive the bonus points.)

Sol. 1: Using param. curve:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-\sqrt{3}\sin t}$$

$$m_{tan} = \frac{dy}{dx}\Big|_{t=\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{6}}{-\sqrt{3}\sin\frac{\pi}{6}} = \frac{2\cdot\frac{\sqrt{3}}{2}}{-\sqrt{3}\cdot\frac{1}{2}}$$

$m_{tan} = -2$

point $(x(\frac{\pi}{6}), y(\frac{\pi}{6})) = (\frac{3}{2}, 1)$

Equ. of tang. line

$\boxed{y-1 = -2\left(x-\frac{3}{2}\right)}$

or $\boxed{y = -2x + 4}$

Sol. 2 Using implicit differentiation

Elliminate the parameter using:

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ so $\frac{x^2}{3} + \frac{y^2}{4} = 1$

Now use implicit differentiation to compute $\frac{dy}{dx}$

$\left(\frac{x^2}{3} + \frac{y^2}{4}\right)' = (1)' \Rightarrow \frac{2x}{3} + \frac{2y}{4}\cdot y' = 0$

$\Rightarrow y' = \frac{dy}{dx} = -\frac{4x}{3y}$

Point $(x(\frac{\pi}{6}), y(\frac{\pi}{6})) = (\frac{3}{2}, 1)$ (as before)

so $m_{tan} = -\frac{4\cdot\frac{3}{2}}{3\cdot 1} = \boxed{-2}$ now use point-slope formula as in Sol. 1

6. (18 pts) Give a complete graph of the function $f(x) = \frac{3x^2-1}{(x-2)^2}$. Your work should include: the domain of the function, equations of eventual asymptotes (vertical or/and horizontal), coordinates of the axis intercepts, a sign chart for the derivative and the second derivative, the location and nature of the critical points (if any), location of inflection points (if any). To save you time, here are the first and the second derivatives $f'(x) = \frac{2(1-6x)}{(x-2)^3}$, $f''(x) = \frac{6(4x+3)}{(x-2)^4}$.

Domain: $x \in \mathbb{R},\ x \neq 2$

$$\lim_{x\to 2^-} \frac{3x^2-1}{(x-2)^2} = \lim_{x\to 2^+} \frac{3x^2-1}{(x-2)^2} = \frac{11}{0^+} = +\infty$$

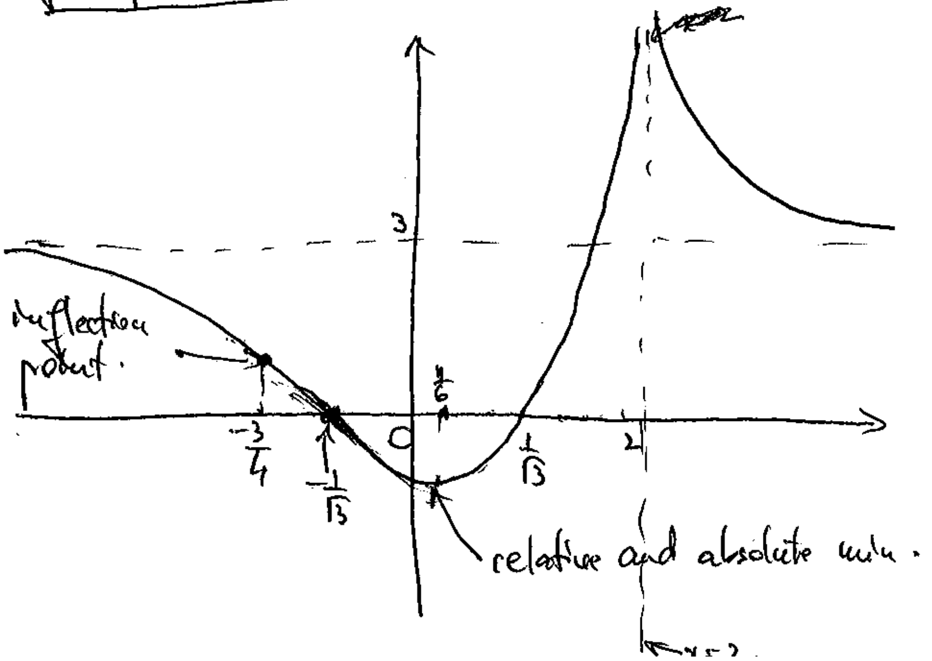
The graph has a vertical asymptote $\underline{x=2}$.

$$\lim_{x\to\pm\infty} \frac{3x^2-1}{(x-2)^2} = \lim_{x\to\pm\infty} \frac{3x^2-1}{x^2-4x+4} = \lim_{x\to\pm\infty} \frac{\cancel{x^2}\left(3-\frac{1}{x^2}\right)}{\cancel{x^2}\left(1-\frac{4}{x}+\frac{4}{x^2}\right)} = 3$$

The graph has horiz. asymptote $y=3$ when $x\to+\infty$ and $x\to-\infty$.

$f'(x) = \frac{2(1-6x)}{(x-2)^3}$. Critical point: $x = \frac{1}{6}$ (note that $x=2$ is not in the domain)

		infl. point			relative min.			
x	$-\infty$	$-\frac{3}{4}$		0	$\frac{1}{6}$		2	$+\infty$
$f'(x)$	$- - - -$	$- - -$	$- - -$	$-$	0	$+ + + +$	‖	$- - - -$
$f(x)$	3 concave down		concave up ↘			↗	‖ $+\infty$	concave up ↘ 3
$f''(x)$	$- - - -$	0	$+ + + +$	$+ + +$	$+$	$+ +$	‖	$+ + + + + + +$



$f\left(\frac{1}{6}\right) = \frac{3\cdot\left(\frac{1}{6}\right)^2-1}{\left(\frac{1}{6}-2\right)^2} < 0.$

y-intercept $f(0) = -\frac{1}{4}$

x-intercepts

$y = f(x) = 0 \Leftrightarrow 3x^2-1=0$

So $x = \pm\frac{1}{\sqrt{3}}$

7. (10 pts) Find the equation of the tangent line to the graph of $f(x) = 3e^{2x}$ at $x = 0$.

Point $(0, f(0))$ so $(0,3)$ (as $e^0 = 1$)

$f'(x) = (3e^{2x})' = 3 \cdot e^{2x} \cdot 2 = 6e^{2x}$

so $m_{tan} = f'(0) = 6$

equ. of tang. line is

$y - 3 = 6(x - 0)$

so $\boxed{y = 6x + 3}$

8. (18 pts) (a) (9 pts) Show that the function $f(x) = x - \sqrt{x}$ satisfies the assumptions of the MVT for $x \in [0, 4]$ and find the value of $c \in (0, 4)$ that satisfies the conclusion of the theorem.

Domain of f: $x \in [0, +\infty)$ f continuous everywhere on its domain so f continuous on $[0,4]$.

$f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}} \Rightarrow f$ differentiable when $x > 0$ so f differentiable on $(0,4)$.

MVT says that there is a point $c \in (0,4)$ so that $f'(c) = \frac{f(4) - f(0)}{4 - 0}$

To find it, solve

$$1 - \frac{1}{2\sqrt{c}} = \frac{4 - \sqrt{4} - 0}{4} = \frac{1}{2} \Rightarrow \frac{1}{2\sqrt{c}} = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow \sqrt{c} = 1 \Rightarrow$$

$$\Rightarrow \boxed{c = 1}$$

(b) (9 pts) Find the absolute minimum and absolute maximum of the function $f(x) = x - \sqrt{x}$ when $x \in [0, 4]$.

Absolute extrema ~~___~~ occur either at a critical point inside the interval or at the end-points.

$f'(x) = 1 - \frac{1}{2\sqrt{x}}$ critical points are $\boxed{x = 0}$ (as $f'(0)$ is not defined)

and $1 - \frac{1}{2\sqrt{x}} = 0 \Rightarrow \frac{1}{2\sqrt{x}} = 1 \Rightarrow \sqrt{x} = \frac{1}{2} \Rightarrow \boxed{x = \frac{1}{4}}$

$f(0) = 0$ $\quad f\left(\frac{1}{4}\right) = \frac{1}{4} - \sqrt{\frac{1}{4}} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ $\quad f(4) = 4 - \sqrt{4} = 2$

Thus absolute min is $\underline{x = \frac{1}{4}}$

absolute max is $\underline{x = 4}$

9. (12 pts) The legs x and y of a right angle triangle vary with time t and so does the hypothenuse z. At a certain moment it is known that $x = 8$ inches and $y = 6$ inches. At that moment, it is also known that x increases at a rate of 1/4 inch/second, while y decreases at a constant rate of 1/2 inch/second. At what rate is the hypothenuse changing at that moment? Is it increasing or decreasing?

$$x^2(t) + y^2(t) = z^2(t) \quad (*)$$

At a moment t_0 we know

$x=8$, $y=6$ so $z = \sqrt{8^2+6^2} = \sqrt{100} = 10$

also $\frac{dx}{dt} = \frac{1}{4}$ $\quad \frac{dy}{dt} = -\frac{1}{2}$. We need to find $\frac{dz}{dt} = ?$

Take derivative of $(*)$ w.r.t. t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

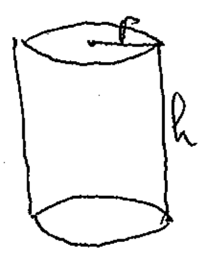
We replace all we know and get

$$2\cdot 8\cdot\frac{1}{4} + 2\cdot 6\left(-\frac{1}{2}\right) = 2\cdot 10\cdot\frac{dz}{dt}$$

$$\boxed{\frac{dz}{dt} = -\frac{1}{10}\ \frac{\text{inch}}{\text{s}}}$$

so at that moment the hypothenuse decreases.

10. (14 pts) A closed cylindrical can is to hold 500 cm^3 of liquid. The material for the top and bottom costs 3 cents per cm^2, while the material for the side costs 2 cents per cm^2. Find the dimensions of the can (radius and height) that will minimize cost.



$$V = \pi r^2\cdot h = 500 \implies h = \frac{500}{\pi r^2}$$

$$\text{Cost} = C = (2\pi r^2)\cdot 3 + (2\pi r h)\cdot 2 = 6\pi r^2 + 4\pi r\cdot h$$

Replace h from the top into the cost formula

$$C(r) = 6\pi r^2 + 4\cancel{\pi} r\cdot\frac{500}{\cancel{\pi} r^2}, \text{ thus } \boxed{C(r) = 6\pi r^2 + \frac{2000}{r}}$$

Domain $r\in(0,+\infty)$

Critical pt.

$$C'(r) = 12\pi r - \frac{2000}{r^2} = 0 \implies 12\pi r^3 = 2000 \implies$$

$$\implies r^3 = \frac{2000}{12\pi} = \frac{1000}{6\pi} \implies r = \frac{10}{\sqrt[3]{6\pi}}$$

Since $\lim_{r\to 0^+} C(r) = +\infty$

and $\lim_{r\to+\infty} C(r) = +\infty$ $\implies r = \frac{10}{\sqrt[3]{6\pi}}$ is a global (absolute) minimum

$$h = \frac{500\,(\sqrt[3]{6\pi})^2}{100\pi} = 5\sqrt[3]{\frac{36}{\pi}}$$