Name: Solution Key Panther ID: ____________

Exam 1 - MAC2311 - Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) These are True or False questions. No justification required. No partial credit. 2 points each.

(i) For all $a > 0, b > 0$, $\log(a^2b^3) = 2\log a + 3\log b$ **True** **False**

(ii) For all $a > 0$, $\sqrt{4a^2+1} = 2a+1$ **True** **False**

(iii) For all $x \neq 0$ $\frac{x}{\sqrt[3]{x^2}} = \frac{1}{\sqrt[3]{x}}$ **True** **False**

(iv) For all $x \neq 0$, $\frac{\sin x}{x} = 1$ **True** **False**

(v) The function $f(x) = \tan x$ is defined and is continuous for all real numbers x. **True** **False**

(vi) If $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$ then $\lim_{x\to 3}(f(x)+2g(x)) = 0$ **True** **False**

2. (6 pts) Consider the function $f(x) = \frac{1}{\sqrt{6-2x}}$.

(a) (3 pts) Find the domain of f. Write your answer in interval form.

$$6-2x > 0 \iff -2x > -6 \iff x < 3 \iff \boxed{x \in (-\infty, 3)}$$

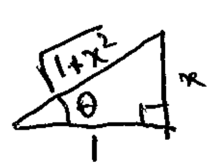
(b) (3 pts) Compute and simplify $f(3-2a^2)$.

$$f(3-2a^2) = \frac{1}{\sqrt{6-2(3-2a^2)}} = \frac{1}{\sqrt{6-6+4a^2}} = \frac{1}{\sqrt{4a^2}} = \frac{1}{2|a|}$$

3. (6 pts) Find an equivalent expression, without inverse trigonometric functions, for $\sec(\arctan x)$.

Let $\arctan x = \theta \iff \tan\theta = x$

We need to determine $\sec\theta = ?$



Pythagora

$$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj.} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$$

Thus $\boxed{\sec(\arctan x) = \sqrt{1+x^2}}$

4. (12 pts) An object is thrown straight up in the air from the ground. Its position $s(t)$ in feet above the ground after t seconds is given by $s(t) = 48t - 16t^2$.

(a) (3 pts) When is the object back on the ground?

$t = ?$ for $s(t) = 0$.

$48t - 16t^2 = 0 \Rightarrow 16t(3-t) = 0 \Rightarrow t = 0$, $\boxed{t = 3s}$

(b) (3 pts) Find the average velocity of the object during the first two seconds of its flight.

$$V_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{32 - 0}{2} = 16 \frac{ft}{s}$$

$s(2) = 48 \cdot 2 - 16 \cdot 2^2 = 32$

$s(0) = 0$

(c) (6 pts) Use limits to find the instantaneous velocity of the object at 2 seconds.

$$V_{inst \text{ at } t=2} = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} \frac{48(2+h) - 16(2+h)^2 - 32}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{96} + 48h - \cancel{64} - 64h - 16h^2 - \cancel{32}}{h} = \lim_{h \to 0} \frac{-16h - 16h^2}{h}$$

$$= \lim_{h \to 0} \frac{-16\cancel{h}(1+h)}{\cancel{h}_1} = \boxed{-16 \frac{ft}{s}}$$

$\leftarrow$ negative, as at $t = 2s$ the object is going down.

5. (12 pts) Given the function below

$$g(x) = \begin{cases} kx^2 - 1 & \text{if } x \le 1 \\ 2x + k & \text{if } x > 1 \end{cases}$$

(a) (6 pts) Is there a value of the constant k which will make the function continuous? Justify your answer.

The only point where $g(x)$ may not be continuous is at $x = 1$.

For $g(x)$ to be continuous at $x = 1$ we must have $\lim_{x\to 1^-} g(x) = \lim_{x \to 1^+} g(x) = g(1)$

But $g(1) = \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (kx^2 - 1) = k - 1$

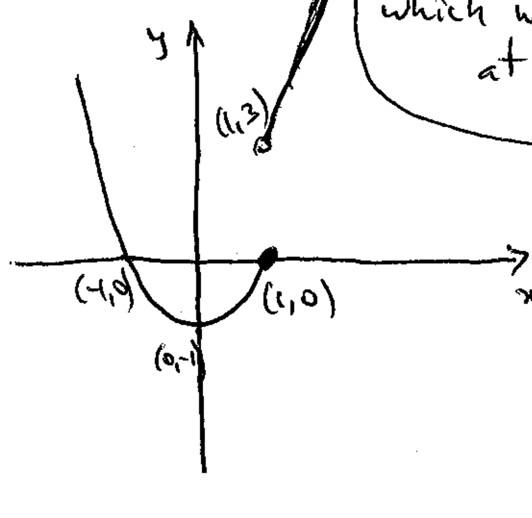
$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (2x + k) = 2 + k$

$\Rightarrow$ Since the equation $k - 1 = 2 + k$ has no solutions, there is no value of k which will make $g(x)$ continuous at $x = 1$

(b) (6 pts) Sketch the graph of the function $g(x)$ when $k = 1$. Label carefully the coordinates of important points.

when $k = 1$

$$g(x) = \begin{cases} x^2 - 1 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$



6. (30 pts) Find the following limits (5 pts each). If the limit is infinite or does not exist, specify so.

(a) $\lim\limits_{x\to1} \dfrac{3x-3}{x^2+2x-3} = \dfrac{0}{0}$

$$= \lim_{x\to1} \frac{3\cancel{(x-1)}}{\cancel{(x-1)}(x+3)} = \boxed{\frac{3}{4}}$$

(b) $\lim\limits_{x\to5^+} \dfrac{1-x}{x-5} = \dfrac{-4}{0^+} = \boxed{-\infty}$

(c) $\lim\limits_{t\to2} \dfrac{|t-2|}{t^2-4} = \lim\limits_{t\to2} \dfrac{|t-2|}{(t-2)(t+2)}$ D.N.E.

$$\lim_{t\to2^+} \frac{|t-2|}{(t-2)(t+2)} = \lim_{t\to2^+} \frac{\cancel{(t-2)}}{\cancel{(t-2)}(t+2)} = \frac{1}{4}$$

$$\lim_{t\to2^-} \frac{|t-2|}{(t-2)(t+2)} = \lim_{t\to2^-} \frac{-(t-2)}{(t-2)(t+2)} = -\frac{1}{4}$$

Thus, the two sided limit

$\lim\limits_{t\to2} \dfrac{|t-2|}{t^2-4}$ D.N.E.

(d) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{x+2} = -\sqrt{2}$

justification

$$\lim_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(1+\frac{2}{x}\right)} = \lim_{x\to-\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(1+\frac{2}{x}\right)} =$$

$$\underset{x<0}{=} \lim_{x\to-\infty} \frac{(-\cancel{x})\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(1+\frac{2}{x}\right)} = \boxed{-\sqrt{2}}$$

(d) $\lim\limits_{x\to0} \dfrac{\sin^2(3x)}{x\tan(2x)} = \dfrac{9}{2}$

justification

$$\lim_{x\to0} \frac{\sin^2(3x)}{x\cdot\tan(2x)} = \lim_{x\to0} \frac{9x^2\cdot\frac{\sin^2(3x)}{(3x)^2}}{x\cdot(2x)\cdot\frac{\tan(2x)}{2x}}$$

$$= \lim_{x\to0} \frac{9\cancel{x^2}\cdot\left(\frac{\sin(3x)}{3x}\right)^2}{2\cancel{x^2}\cdot\frac{\tan(2x)}{2x}} = \boxed{\frac{9}{2}}$$

(e) $\lim\limits_{x\to+\infty} \dfrac{\cos(3x)}{x} = 0$

justification $\to$ Squeeze Theorem.

$-1 \le \cos(3x) \le 1 \quad \div x \quad (x>0)$

$$-\frac{1}{x} \le \frac{\cos(3x)}{x} \le \frac{1}{x}$$

$x\to+\infty$ $\qquad$ $x\to+\infty$ $\qquad$ $x\to+\infty$

0

7. (10 pts) Sketch the graph of a function $f(x)$ satisfying all of the following conditions.

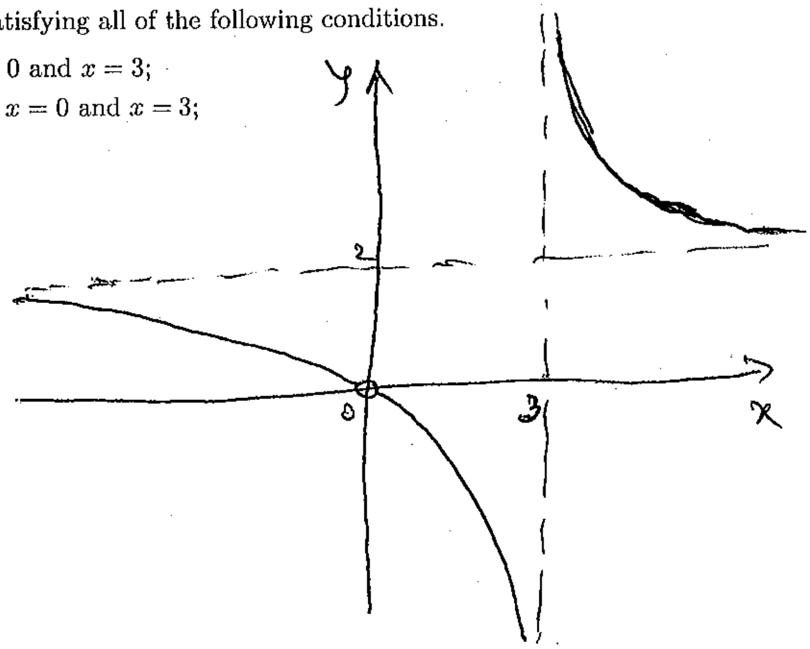
(i) The function is defined everywhere except $x = 0$ and $x = 3$;

(ii) The function is continuous everywhere except $x = 0$ and $x = 3$;

(iii) $\lim_{x\to 0} f(x) = 0$;

(iv) $\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = +\infty$;

(v) $\lim_{x\to -\infty} f(x) = 2$ and $\lim_{x\to +\infty} f(x) = 2$;



Lots of other graphs are possible, but for all $x=3$ should be a vertical asymptote $y=2$ should be a horiz. asymptote and there should be a removable discontinuity at $(0,0)$.

Bonus 5 pts: Find a formula for a function $f(x)$ satisfying all conditions (i)-(v) above.

$$f(x) = \frac{2x^2}{x(x-3)}$$

8. (10 pts) (a) (3 pts) Write the general (ϵ, δ) definition for $\lim_{x\to a} f(x) = L$.

Given $\varepsilon > 0$, we can find $\delta > 0$ so that if $|x-a| < \delta$, $x \neq a$, then $|f(x) - L| < \varepsilon$.

(b) (7 pts) Use the (ϵ, δ) definition to prove $\lim_{x\to -3}(10x+3) = -27$.

$$|f(x)-L| = |10x+3-(-27)| = |10x+30| = 10|x+3|$$

Thus $|f(x) - L| < \varepsilon \iff 10|x+3| < \varepsilon \iff |x+3| < \frac{\varepsilon}{10}$

Thus given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{10}$.

Then if $|x-(-3)| < \delta \Rightarrow |f(x)-L| = 10|x+3| < 10\delta = \cancel{10}\frac{\varepsilon}{\cancel{10}} = \varepsilon$

9. (10 pts) Choose ONE of the following:

(a) State and prove the quadratic formula.

(b) Prove the inequality $\sin x \leq x \leq \tan x$ for any $x \in [0, \pi/2)$.

(c) Use the Intermediate Value Theorem to show that the equation $x^3 = 4x - 1$ has three real solutions. Locate these solutions in intervals of length one.

For (a), (b) see your notes or text book

Solution for (c)

$x^3 = 4x - 1 \iff x^3 - 4x + 1 = 0$

Let $f(x) = x^3 - 4x + 1$ continuous everywhere (polynomial)

$f(0) = 1 > 0$
$f(1) = -2 < 0$ $\overset{\text{By I.V.T.}}{\Longrightarrow}$ there is a point $c_1 \in (0,1)$ so that $f(c_1) = 0$.

$f(2) = 1 > 0$ $\overset{\text{By I.V.T.}}{\Longrightarrow}$ there is a point $c_2 \in (1,2)$ so that $f(c_2) = 0$.

$f(0) = 1 > 0$
$f(-1) = 4 > 0$
$f(-2) = 1 > 0$
$f(-3) = -14 < 0$ $\overset{\text{I.V.T.}}{\Longrightarrow}$ there is a point $c_3 \in (-3, 2)$ so that $f(c_3) = 0$.

Thus the equation has three real solutions, one in each of the intervals $(0,1)$, $(1,2)$, $(-3,-2)$.