NAME: Solution Key

Panther ID: ____________

Exam 2 - MAC 2311 Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (30 pts) Find dy/dx. Simplify when possible (6 pts each):

(a) $y = \frac{x^3}{3} - 2\sqrt{x} + 10^6 = \frac{1}{3}x^3 - 2x^{\frac{1}{2}} + \underbrace{10^6}_{\text{constant}}$

$y' = \frac{1}{\cancel{3}} \cdot \cancel{3}x^2 - \cancel{2} \cdot \frac{1}{\cancel{2}} x^{-\frac{1}{2}} + 0$

$\boxed{y' = x^2 - \frac{1}{\sqrt{x}}}$

(b) $y = e^{3x}\sec x$ Use Product Rule

$y' = (e^{3x})' \sec x + e^{3x}(\sec x)'$

$y' = e^{3x} \cdot 3 \cdot \sec x + e^{3x} \sec x \cdot \tan x$

$y' = e^{3x} \cdot \sec x \, (3 + \tan x)$

(c) $y = \frac{1}{2x + \sin^3 x} = (2x + \sin^3 x)^{-1}$

$y' = (-1)(2x + \sin^3 x)^{-2} \cdot (2x + \sin^3 x)'$

$y' = -\frac{1}{(2x + \sin^3 x)^2} \cdot (2 + 3\sin^2 x \cdot \cos x)$

$\boxed{y' = -\frac{2 + 3\sin^2 x \cdot \cos x}{(2x + \sin^3 x)^2}}$

Solution with Quotient Rule also OK

(d) $y = \arcsin(\cos x)$

Sol. 1 (Chain Rule) $(\arcsin(u(x)))' = \frac{1}{\sqrt{1 - u(x)^2}} \cdot u'(x)$

$y' = \frac{1}{\sqrt{1 - \cos^2 x}} \cdot (\cos x)'$

$y' = \frac{1}{\sin x} \cdot (-\sin x) = \boxed{-1}$

Sol. 2. Use the identity (explained in class)

$\arcsin z + \arccos z = \frac{\pi}{2}$

$y = \arcsin(\cos x) = \frac{\pi}{2} - \arccos(\cos x)$

so $y = \frac{\pi}{2} - x$. Thus $y' = \left(\frac{\pi}{2} - x\right)' = -1$

Sol. 3. with Implicit diff.

$y = \arcsin(\cos x) \iff \sin y = \cos x$

Thus $\frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$

$(\cos y)\, y' = -\sin x \Rightarrow y' = -\frac{\sin x}{\cos y} = -\frac{\sin x}{\sqrt{1 - \sin^2 y}} =$

$\frac{\sin x}{\sin x}$

(e) $y = (\ln x)^x$ Log. differentiation

$\ln y = \ln((\ln x)^x) = x \cdot \ln(\ln x)$

$(\ln y)' = (x \cdot \ln(\ln x))'$

$\frac{1}{y} \cdot y' = 1 \cdot \ln(\ln x) + \cancel{x} \cdot \frac{1}{\ln x} \cdot \frac{1}{\cancel{x}}$

$y' = y \cdot \left(\ln(\ln x) + \frac{1}{\ln x}\right)$

$y' = (\ln x)^x \cdot \left(\ln(\ln x) + \frac{1}{\ln x}\right)$

2. (8 pts) If $f(x) = \sin(2x)$, determine $f^{(2014)}(x)$.

We need to find a pattern:

$$f^{(0)}(x) = f(x) = \sin(2x)$$
$$f^{(1)}(x) = f'(x) = (\sin(2x))' = \cos(2x) \cdot 2$$
$$f^{(2)}(x) = f''(x) = (f'(x))' = (2\cos(2x))' = -2\cancel{\sin}(2x) \cdot 2 = -2^2 \sin(2x)$$
$$f^{(3)}(x) = (f^{(2)}(x))' = -2^3 \cos(2x)$$
$$f^{(4)}(x) = (f^{(3)}(x))' = 2^4 \sin(2x)$$

So, the pattern is that the powers of 2 increase, while the "sin, cos, -sin, -cos" repeats

Since $2014 = 2012 + 2 = 4 \times 503 + 2$

in $f^{(2014)}(x)$ we'll have $-\sin(2x)$, as in $f^{(2)}(x)$

Thus $\boxed{f^{(2014)}(x) = -2^{2014} \sin(2x)}$

3. (10 pts) The function $h(x)$ is given by $h(x) = \frac{f(x)}{1+x^2}$. Given that $f(2) = 5$ and $f'(2) = 1$, find

(a) (3 pts) $h(2)$

$$h(2) = \frac{f(2)}{1+2^2} = \frac{5}{5} = 1$$

(b) (7 pts) $h'(2)$

$$h'(x) = \left(\frac{f(x)}{1+x^2}\right)' = \frac{f'(x)(1+x^2) - f(x) \cdot 2x}{(1+x^2)^2}$$

$$h'(2) = \frac{f'(2)(1+2^2) - f(2) \cdot 2 \cdot 2}{(1+2^2)^2}$$

$$h'(2) = \frac{1 \times 5 - 4 \times 5}{5^2} = \frac{-15}{25} = \boxed{-\frac{3}{5}}$$

4. (12 pts) Find the equation of the tangent line to the curve $3x - x^2y^2 = 2y^3$ at the point $(1, 1)$.

We need $m_{tan} = \frac{dy}{dx}\Big|_{(1,1)}$

To find $\frac{dy}{dx}$ we use implicit differentiation

$$(3x - x^2y^2)' = (2y^3)'$$

$$3 - \left[(x^2)'y^2 + x^2(y^2)'\right] = 6y^2 \cdot y'$$

$$3 - 2x \cdot y^2 - 2x^2 y \cdot y' = 6y^2 \cdot y'$$

$$3 - 2xy^2 = 2x^2y \cdot y' + 6y^2 \cdot y'$$

$$3 - 2xy^2 = y'(2x^2y + 6y^2) \Rightarrow y' = \frac{dy}{dx} = \frac{3-2xy^2}{2x^2y + 6y^2}$$

$$m_{tan} = \frac{dy}{dx}\Big|_{(1,1)} = \frac{3-2}{2+6} = \frac{1}{8}$$

Thus, the tangent line is $\boxed{y - 1 = +\frac{1}{8}(x-1)}$

5. (14 pts) (a) (8 pts) Find the local linear approximation of the function $f(x) = \tan x$ at $x_0 = \pi/4$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = (\tan x)' = \sec^2 x$$

$$\text{so } f'(x_0) = f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \boxed{2}$$

$$\boxed{\tan x \approx 1 + 2\left(x - \frac{\pi}{4}\right)}$$

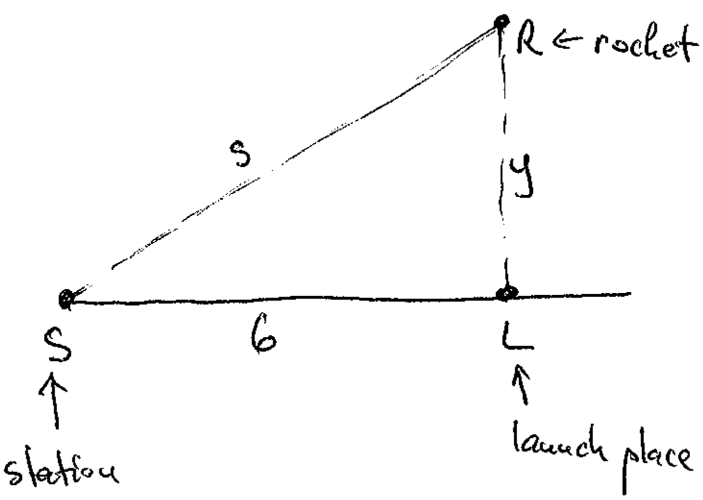
(b) (6 pts) Use the result of part (a) to estimate without calculator $\tan 47°$. (OK for your answer to contain π.)

$47° \leftrightarrow 47 \cdot \frac{\pi}{180}$ radians

In the above approximation formula take $x = \frac{47\pi}{180}$

$$\tan\left(\frac{47\pi}{180}\right) \approx 1 + 2\left(\frac{47\pi}{180} - \frac{\pi}{4}\right) \quad \text{or} \quad \tan(47°) = \tan\left(\frac{47\pi}{180}\right) \approx 1 + \frac{\pi}{45}$$ (after a bit of algebra)

6. (12 pts) A rocket that is launched vertically is tracked by a radar station located on the ground 6 miles from the launch site. What is the vertical speed of the rocket at the instant its distance from the radar station is 10 miles and this distance increases at the rate of 3600 mi/h?



In the picture, let

$s = |SR|$ ← the distance from the rocket to the radar station

$y = |LR|$ ← the height of the rocket above ground measured from the launch site

Note that

$s = s(t)$, $y = y(t)$ both vary with time t

Note also that the distance $|SL| = 6 =$ const

We know that when $s = 10$, $\frac{ds}{dt} = 3600\,\frac{mi}{h}$. We need to find $\frac{dy}{dt}$.

Pythagora $\Rightarrow$ $6^2 + y^2 = s^2$. Take $\frac{d}{dt}$ of both sides

Get $0 + 2y\frac{dy}{dt} = 2s\frac{ds}{dt} \Rightarrow \boxed{\frac{dy}{dt} = \frac{s}{y} \cdot \frac{ds}{dt}}$

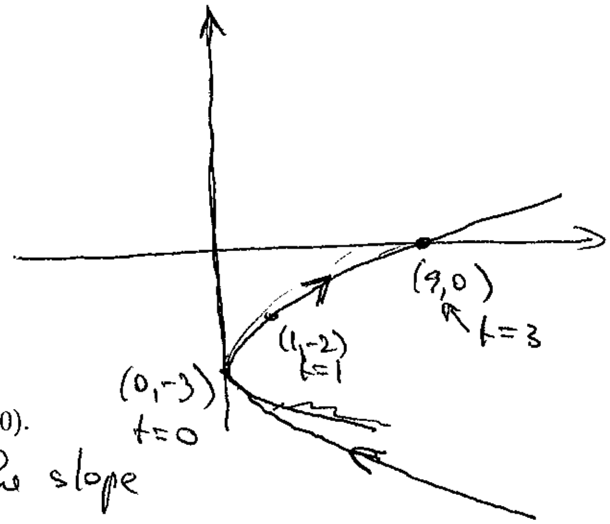
when $s = 10$, $y = \sqrt{10^2 - 6^2} = 8$ so $\left.\frac{dy}{dt}\right|_{\text{when } s=10} = \frac{10}{\cancel{8}_2} \cdot \cancel{3600}^{900} = \frac{9000}{2} = \boxed{4500\,\frac{mi}{h}}$

7. (12 pts) Given the parametric curve $x = t^2$, $y = t - 3$:

(a) (4 pts) Sketch the curve in the xy plane, clearly indicating orientation.

Eliminating the parameter

$t = y+3$ so $x = (y+3)^2$



(b) (8 pts) Find the tangent line to the curve at the point $(9, 0)$.

The point is (9,0), we need to find the slope of the tangent

$$m_{tan} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2t}$$ ← this is the slope at an arbitrary value t of the parameter

But the point (9,0) is obtained for $t = 3$ (Solve $9 = t^2$, $0 = t-3$)

So $m_{tan} = \frac{1}{2\cdot 3} = \frac{1}{6}$. Tang. line is $y - 0 = \frac{1}{6}(x-9)$

$$\boxed{y = \frac{x}{6} - \frac{3}{2}}$$

8. (12 pts) Choose ONE:

(a) State and prove the formula for the derivative of a product of two functions.

(b) Use the limit definition of the derivative to show that $(\sin x)' = \cos x$.

See textbook or your notes