NAME: Solution Key Panther ID: ____________

Exam 3 - MAC 2311 Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (16 pts) Compute each of the following limits:

(a) $\lim_{x\to 0} \frac{\arctan(2x)}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}}$

$$= \lim_{x\to 0} \frac{\frac{1}{1+(2x)^2}\cdot 2}{1} = \boxed{2}$$

Alternative solution using substitution:

let $\theta = \arctan(2x)$

$\Rightarrow \tan\theta = 2x \Rightarrow x = \frac{\tan\theta}{2}$

$$\lim_{x\to 0}\frac{\arctan(2x)}{x} = \lim_{\theta\to 0}\frac{\theta}{\frac{\tan\theta}{2}}$$

$$= \lim_{\theta\to 0} 2\cdot\frac{\theta}{\tan\theta} = 2\cdot 1 = \boxed{2}$$

(b) $\lim_{x\to+\infty}(1-3/x)^{2x} =$ 1^{∞} ← power indeterminate form

$$= \lim_{x\to+\infty} e^{\ln\left(\left(1-\frac{3}{x}\right)^{2x}\right)} =$$

$$= \lim_{x\to+\infty} e^{2x\cdot\ln\left(1-\frac{3}{x}\right)} = e^{\lim_{x\to+\infty} 2x\cdot\ln\left(1-\frac{3}{x}\right)} = *$$

$$\lim_{x\to+\infty} 2x\cdot\ln\left(1-\frac{3}{x}\right) \overset{\infty\cdot 0}{=} 2\cdot\lim_{x\to+\infty}\frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}} =$$

$$\underset{\substack{\text{sub.}\\ w=\frac{1}{x}}}{=} 2\cdot\lim_{w\to 0^+}\frac{\ln(1-3w)}{w} \overset{\frac{0}{0}}{\underset{l'H}{=}}$$

note $x\to+\infty \Rightarrow w\to 0^+$

$$= 2\cdot\lim_{w\to 0^+}\frac{\frac{1}{1-3w}\cdot(-3)}{1} = -6$$

Thus $* = \boxed{e^{-6}}$

2. (8 pts) True or False questions. No justification needed. 2 points each.

(a) If $f'(x_0) = 0$ then x_0 is relative maximum or a relative minimum for the function $f(x)$. **True** **False**

(b) If $f'(2) = 0$ and $f''(2) > 0$ then f has a relative minimum at $x = 2$. **True** **False**

(c) If $f'(x) < 0$ for all $x \in [a, b]$, then $x = a$ is an absolute maximum for $f(x)$ on the interval $[a, b]$. **True** **False**

(d) If $f''(x) = 0$ for all $x \in \mathbf{R}$, then $f(x) = mx + b$ for some constants m and b. **True** **False**

3. (24 pts) Find the indicated antiderivatives:

(a) $\int \left(\sec^2 x - \frac{1}{1+x^2}\right) dx =$

$= \tan x - \arctan x + C$

(b) $\int \frac{3+x^2}{2x}\, dx = \int \left(\frac{3}{2x} + \frac{x^2}{2x}\right) dx =$

$= \int \left(\frac{3}{2} \cdot \frac{1}{x} + \frac{x}{2}\right) dx = \frac{3}{2} \ln x + \frac{x^2}{4} + C$

(c) $\int x^3 \sqrt{x^4+1}\, dx =$

sub $w = x^4 + 1$

$dw = 4x^3 dx$

$\frac{1}{4} dw = x^3 dx$

$= \int \sqrt{w}\, \frac{1}{4} dw =$

$= \frac{1}{4} \int w^{\frac{1}{2}} dw$

$= \frac{1}{4} \cdot \frac{2}{3} w^{\frac{3}{2}} + C$

$= \boxed{\frac{1}{6} (x^4+1)^{\frac{3}{2}} + C}$

(d) $\int \frac{1}{x(\ln x)^2}\, dx =$

sub $w = \ln x$

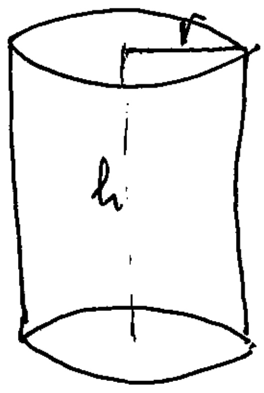
$dw = \frac{1}{x} dx$

$= \int \frac{1}{w^2} dw = \int w^{-2} dw$

$= -w^{-1} + C = -\frac{1}{w} + C$

$= \boxed{-\frac{1}{\ln x} + C}$

5. (14 pts) You are asked to make a cylindrical can with a given volume of 81π cm^3. The top and the bottom of the can should be made from a material that costs 3 cents per cm^2, while the side of the can should be made from a material that costs 2 cents per cm^2. Find the dimensions of the can (radius and height) that will minimize the cost.



Let C be the total cost of the can

$$C = Cost_{top+bottom} + C_{side}$$

$$C = 3\cdot(A_{top+bottom}) + 2\cdot A_{side}$$

unit costs per cm^2.

$$A_{top+bottom} = \pi r^2 + \pi r^2 = 2\pi r^2$$

$$A_{side} = (2\pi r)\cdot h$$ (unfold the side to get a rectangle with width $2\pi r$ - circumference of base, and height h)

thus $C = 3\cdot 2\pi r^2 + 2\cdot(2\pi r)\cdot h = 6\pi r^2 + 4\pi r h$

$V_{cyl.} = \pi r^2\cdot h = 81\pi$ (given) $\Rightarrow h = \frac{81}{r^2}$ substitute

$$C(r) = 6\pi r^2 + 4\pi r\cdot\frac{81}{r^2} = 6\pi\left(r^2 + \frac{54}{r}\right), \quad r\in(0,+\infty)$$

We want to find abs. minimum of $C(r)$

$$C'(r) = 6\pi\left(2r - \frac{54}{r^2}\right)$$

$C'(r) = 0 \iff 2r = \frac{54}{r^2} \iff r^3 = 27 \iff r = 3$ the only critical point

Since $\lim_{r\to 0^+} C(r) = \lim_{r\to+\infty} C(r) = +\infty$, $r=3$ is the abs. minimum of $C(r)$.

So for minimum cost, $\boxed{r = 3\text{ cm},\ h = \frac{81}{3^2} = 9\text{ cm}}$.

6. (12 pts) (a) (4 pts) State the Mean Value Theorem.

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is (at least) one point $c \in (a,b)$ so that
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(b) (8 pts) Verify that the hypothesis of the Mean Value Theorem are satisfied for the function $f(x) = 1/x$ on the interval $[1, 4]$, and find the value(s) of $c \in (1, 4)$ that satisfy the conclusion of the Theorem.

The domain of $f(x) = \frac{1}{x}$ is $(-\infty, 0) \cup (0, +\infty)$ and f is continuous and differentiable at all points in the domain. In particular, f is continuous and differentiable on $[1,4]$, so MVT applies.

To find c, $f'(x) = -\frac{1}{x^2}$ and $\frac{f(4)-f(1)}{4-1} = \frac{\frac{1}{4}-1}{4-1} = \frac{-\frac{3}{4}}{3} = -\frac{3}{4}\cdot\frac{1}{3} = -\frac{1}{4}$

So $-\frac{1}{c^2} = -\frac{1}{4} \Rightarrow c^2 = 4 \Rightarrow c = \pm 2$

$c=-2$ is not in the interval $(1,4)$, but $c=2 \in (1,4)$. Thus $\boxed{c=2}$ satisfies MVT for $f(x)=\frac{1}{x}$ on $[1,4]$.

7. (14 pts) (a) (8 pts) Suppose an object is moving on a straight line with constant acceleration a. Use integration to find the formulas for the velocity $v(t)$ and the position $s(t)$ of the object at time t.

(b) (6 pts) A baseball is thrown straight upward from ground level with an initial velocity of 96 ft/s. Does the baseball reach the top of a building 160 ft tall? Use part (a) to justify your answer. Assume gravitational acceleration $g = -32$ ft/s^2.

(a) $a(t) = a \Rightarrow v(t) = \int a\,dt = at + c$

$v(0) = 0 + c \Rightarrow c = v(0) = v_0$ $\Rightarrow \boxed{v(t) = at + v_0}$

$s(t) = \int v(t)\,dt = \int (at + v_0)\,dt = \frac{at^2}{2} + v_0 t + \tilde{c}$

at $t=0$ $s(0) = 0 + 0 + \tilde{c} \Rightarrow \tilde{c} = s(0) = s_0$.

Thus, $\boxed{s(t) = \frac{at^2}{2} + v_0 t + s_0}$

(b) $a = g = -32$, $v_0 = 96$, $s_0 = 0$ (initially, ball is at ground level)

Maximum height is attained when $s'(t) = v(t) = 0$

so $-32t + 96 = 0 \Rightarrow$ at $t = 3s$ the ball has max. height.

$s(3) = -16 \times 3^2 + 96 \times 3 = 144$ ft. Thus the ball does not reach 160 ft.

Alternative sol: Use quadratic formula to show equation $160 = -16t^2 + 96t$ has no real solutions

8. (20 pts) The steps of this problem should lead you to a complete graph of the function $f(x) = x^2e^{-x}$. Where indicated, work should be shown below, or on a separate sheet of paper.

(a) (2 pts) The domain of this function is $x \in (-\infty, +\infty)$

(b) (3 pts) The derivative, in factored form, is $f'(x) = e^{-x} \cdot x \cdot (2-x)$. Show work.

$f'(x) = 2xe^{-x} + x^2e^{-x} \cdot (-1) \leftarrow$ chain Rule

(c) (3 pts) Critical points of f (if any): $x =$ $\boxed{x=0}, \boxed{x=2}$. Show work.

(d) (3 pts) Do a sign chart for f' and mark the intervals where f is increasing, respectively decreasing.

x	$-\infty$		0		2		$+\infty$
$f'(x)$		$-\ -\ -\ -\ -\ -$	0	$+\ +\ +$	0	$-\ -\ -\ -$	
$f(x)$	$+\infty$	$\longrightarrow$	0	$\nearrow$	$4e^2$	$\longrightarrow$	0

(e) (4 pts) End behavior: $\lim\limits_{x\to-\infty} x^2e^{-x} = \infty \cdot \infty = +\infty$ $\qquad \lim\limits_{x\to+\infty} x^2e^{-x} =$ $\infty \cdot 0$ Show work below .

$= \lim\limits_{x\to+\infty} \dfrac{x^2}{e^x} \underset{\text{l'H twice}}{=} 0$

(f) (5 pts) Using all the previous steps, sketch the graph of $f(x) = x^2e^{-x}$. Label on the graph the coordinates of critical points (if any) and also specify the type of the critical point.

Bonus 2pts: I did not ask you to do the analysis of the second derivative. Without computing the second derivative, how many inflection points do you expect? 2 $\leftarrow$ this can be confirmed with a sign chart for $f''(x)$