Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus I Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Turn off your cell phone at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (24 pts) Compute y' (6 pts each):

(a) $y = e^x + x^\pi - \pi^e$

$$y' = e^x + \pi \cdot x^{\pi - 1} \qquad (\pi^e \text{ is a } \underline{\text{constant}})$$

(b) $y = \dfrac{\sec(3x)}{2x+1}$

$$y' = \left(\frac{\sec(3x)}{2x+1}\right)' \overset{\text{Q. Rule}}{=} \frac{(\sec(3x))'(2x+1) - \sec(3x)\cdot(2x+1)'}{(2x+1)^2} =$$

$$= \frac{\sec(3x)\cdot\tan(3x)\cdot 3\cdot(2x+1) - 2\sec(3x)}{(2x+1)^2} = \frac{\sec(3x)\left[3(2x+1)\tan(3x) - 2\right]}{(2x+1)^2}$$

(c) $y = \dfrac{e^{5x}\cdot\sqrt[3]{x}}{x^2+1}$ (only logarithmic differentiation is acceptable for this one)

$$\ln y = \ln\left(\frac{e^{5x}\cdot\sqrt[3]{x}}{x^2+1}\right) = \ln(e^{5x}) + \ln(x^{\frac{1}{3}}) - \ln(x^2+1)$$

$$\ln y = 5x + \frac{1}{3}\ln x - \ln(x^2+1) \quad \Big|\ \frac{d}{dx} \text{ to both sides}$$

$$\frac{1}{y}\cdot y' = 5 + \frac{1}{3}\cdot\frac{1}{x} - \frac{2x}{x^2+1} \rightarrow \boxed{y' = \frac{e^{5x}\cdot\sqrt[3]{x}}{x^2+1}\cdot\left[5 + \frac{1}{3x} - \frac{2x}{x^2+1}\right]}$$

(d) $y = \sqrt{1+\cos^4(5x)}$

$$y' = \left(\left(1+\cos^4(5x)\right)^{\frac{1}{2}}\right)' = \frac{1}{2}\left(1+\cos^4(5x)\right)^{-\frac{1}{2}}\cdot\left(1+\cos^4(5x)\right)'$$

$$y' = \frac{1}{2}\left(1+\cos^4(5x)\right)^{-\frac{1}{2}}\cdot 4\cos^3(5x)\cdot\left(\cos(5x)\right)'$$

$$y' = \frac{1}{\cancel{2}}\left(1+\cos^4(5x)\right)^{-\frac{1}{2}}\cdot \overset{2}{\cancel{4}}\cos^3(5x)\cdot(-\sin(5x))\cdot 5 = \boxed{-\frac{10\cos^3(5x)\sin(5x)}{\sqrt{1+\cos^4(5x)}}}$$

2. (10 pts) Use implicit differentiation to find the tangent line to the curve $e^{x^2-y^2} = xy$ at $(1,1)$.

$$\frac{d}{dx}\left(e^{x^2-y^2}\right) = \frac{d}{dx}(x\cdot y)$$

$$e^{x^2-y^2}\cdot(2x-2y\cdot y') = 1\cdot y + x\cdot y'$$

$$2xe^{x^2-y^2} - 2y\cdot e^{x^2-y^2}\cdot y' = y + x\cdot y'$$

$$2xe^{x^2-y^2} - y = \left(2ye^{x^2-y^2}+x\right)\cdot y'$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{2xe^{x^2-y^2}-y}{2ye^{x^2-y^2}+x}$$

$$\Rightarrow m_{tan} = \frac{dy}{dx}\Big|_{(1,1)} = \frac{2\cdot1\cdot e^0-1}{2\cdot1 e^0+1} = \frac{1}{3}$$

Easier Solution: First apply ln to both side

$$\ln\left(e^{x^2-y^2}\right) = \ln(x\cdot y)$$

$$x^2-y^2 = \ln x + \ln y \leftarrow \text{Apply } \frac{d}{dx}$$

$$\frac{d}{dx}(x^2-y^2) = \frac{d}{dx}(\ln x + \ln y)$$

$$2x - 2y\cdot y' = \frac{1}{x} + \frac{1}{y}\cdot y'$$

$$2x - \frac{1}{x} = 2y\cdot y' + \frac{1}{y}\cdot y'$$

$$y' = \frac{2x-\frac{1}{x}}{2y+\frac{1}{y}} \Rightarrow m_{tan} = \frac{dy}{dx}\Big|_{(1,1)} = \frac{2\cdot1-\frac{1}{1}}{2\cdot1+\frac{1}{1}} = \boxed{\frac{1}{3}}$$

equation of tang. line is $\boxed{y-1 = \frac{1}{3}(x-1)}$

3. (24 pts) Compute the following limits, SHOWING YOUR WORK. If the limit does not exist or is infinite specify so.

(a) $\boxed{\lim_{x\to 2}\frac{x^2-4}{|x-2|}}$ Must compute the one-sided limits

D.N.E.

$$\lim_{x\to2^+}\frac{x^2-4}{|x-2|} = \lim_{x\to2^+}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}\,1} = 4$$

$\neq$

$$\lim_{x\to2^-}\frac{x^2-4}{|x-2|} = \lim_{x\to2^-}\frac{(x-2)(x+2)}{-(x-2)} = -4$$

(b) $\lim_{x\to-\infty}\frac{2x^5-3x^2+7x-2}{x^4+x^2+1} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to-\infty}\frac{x^{\cancel{5}}\left(2-\frac{3}{x^3}+\frac{7}{x^4}-\frac{2}{x^5}\right)}{\cancel{x^4}\left(1+\frac{2}{x^2}+\frac{1}{x^4}\right)}$

$$= \frac{-\infty\cdot(2)}{1} = -\infty$$

Using "junk rule" is also acceptable (or l'Hopital)

(c) $\lim_{x\to+\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to+\infty}\frac{\ln x}{x^{\frac{1}{2}}} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}}$

$$= \lim_{x\to+\infty}\frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to+\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} =$$

$$= \lim_{x\to+\infty}\frac{2\sqrt{x}}{x} = \lim_{x\to+\infty}\frac{2}{\sqrt{x}} = \boxed{0}$$

(d) $\lim_{x\to0}(\cos x)^{1/x^2}$ $\quad 1^\infty$ use $A = e^{\ln A}$ trick

$$= \lim_{x\to0} e^{\ln\left[(\cos x)^{\frac{1}{x^2}}\right]} = e^{\lim_{x\to0}\frac{\ln(\cos x)}{x^2}} = *$$

$$\lim_{x\to0}\frac{\ln(\cos x)}{x^2} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to0}\frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x} =$$

$$= \lim_{x\to0}\left(-\frac{1}{2}\cdot\frac{1}{\cos x}\cdot\frac{\sin x}{x}\right) = -\frac{1}{2}\cdot\frac{1}{\cos 0}\cdot 1 = -\frac{1}{2}$$

Thus, $\boxed{* = e^{-\frac{1}{2}}}$

4. (20 pts) Find each indicated antiderivative:

(a) $\int \left(3\cos x + \frac{2}{x} - \frac{1}{\sqrt{1-x^2}}\right) dx = 3\sin x + 2\ln x - \arcsin x + c$

Note about (b): The two apparently different answers are equivalent (up to a constant) because $\sec^2 x = \tan^2 x + 1$

(b) $\int \sec^2 x \tan x \, dx =$

$\boxed{\text{Sol. 1}}$ subst. $u = \tan x$

$du = \sec^2 x \, dx$

$\int u \, du = \frac{u^2}{2} + c$

$\boxed{= \frac{(\tan x)^2}{2} + c}$

or $\boxed{\text{Sol. 2}}$ $= \int \sec x \cdot \sec x \cdot \tan x \, dx =$

sub. $w = \sec x$

$dw = \sec x \cdot \tan x \, dx$

$= \int w \, dw = \frac{w^2}{2} + c$

$= \boxed{\frac{(\sec x)^2}{2} + c}$

(c) $\int x^2\sqrt{x^3+4}\, dx = \int \sqrt{u} \cdot \frac{1}{3} du =$

subst. $u = x^3 + 4$

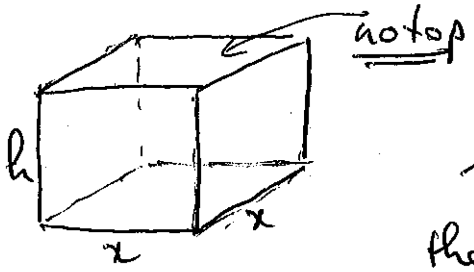
$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$= \frac{1}{3}\int u^{\frac{1}{2}} du = \frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} + c$

$\boxed{= \frac{2}{9}\left(x^3+4\right)^{\frac{3}{2}}}$

5. (14 pts) Using calculus, find the dimensions of an open rectangular packaging box (with no top), made of cardboard, that satisfies all of the following requirements: (a) the volume of the box is 1000 cm^3; (b) the base is a square; (c) the amount of cardboard used to make the box should be minimal.



$V = x^2 \cdot h = 1000$ $\quad$ x = length of base, h = height

$\Rightarrow h = \frac{1000}{x^2}$

The amount of cardboard use corresponds to the total surface area S of the box.

We want, thus, to minimize S.

$S = \underbrace{x^2}_{\text{area of base}} + \underbrace{4x \cdot h}_{\text{area of the 4 lateral sides}} \underset{\text{use } h = \frac{1000}{x^2}}{=} x^2 + 4x \cdot \frac{1000}{x^2} = x^2 + \frac{4000}{x}$

Thus we minimize $S(x) = x^2 + \frac{4000}{x}$ when $x \in (0, +\infty)$

$S'(x) = 2x - \frac{4000}{x^2}$, so $S'(x) = 0 \Leftrightarrow 2x = \frac{4000}{x^2} \Leftrightarrow x^3 = 2000 \Leftrightarrow$

$\Rightarrow \boxed{x = \sqrt[3]{2000} = 10\sqrt[3]{2}}$ is the only critical point

It is the absolute min. because $\lim_{x \to 0^+} S(x) = \lim_{x \to +\infty} S(x) = +\infty$.

Thus $x = 10\sqrt[3]{2}$ cm

$h = \frac{1000}{100\sqrt[3]{4}} = 10\sqrt[3]{4}$ c

6. (12 pts) These are True or False questions. No partial credit. 2 points each.

a. If $\lim_{x \to 2} f(x) = f(2)$, then f is continuous at $x = 2$. **True** **False**

b. If $f'(2) = 0$, then f has a relative maximum or minimum at $x = 2$. **True** **False**

c. To compute the derivative of $\sec(\tan x)$ we must use the product rule. **True** **False**

d. If f is continuous at $x = 2$, then f is differentiable at $x = 2$. **True** **False**

e. If $\lim_{x\to 2} f(x) = f(2) = 5$, then for x sufficiently close to 2, $4.99 < f(x) < 5.01$. **True** **False**

f. If $\lim_{x\to 2} f(x) = f(2) = 5$, then for x sufficiently close to 2, $f(x) \neq 5.1$. **True** **False**

7. (12 pts) For each of the following, fill in the blanks with the most appropriate words or expressions:

(a) The definition with limit of the derivative is $f'(x) =$ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on the interval (a, b).

(c) The average rate of change of a function $f(x)$ on an interval $[a, b]$, is given by $\frac{f(b)-f(a)}{b-a}$.

(d) The derivative with respect to time of velocity is acceleration, $a(t)$.

(e) A polynomial function of degree 5 can have at most 4 critical points.

(f) If $f'(5) = 0$ and $f''(5) > 0$, then x_0 is a relative min (local min) for the function.

8. (10 pts) (a) (6 pts) Find the local linear approximation of the function $f(x) = (1 + x)^7$ at $x_0 = 0$.

Loc. lin. approx. formula $f(x) \approx f(x_0) + f'(x_0)(x-x_0)$

$f'(x) = ((1+x)^7)' = 7(1+x)^6 \cdot 1$

Thus $f'(x_0) = f(0) = 7 \cdot 1^6 = 7$. Also $f(x_0) = f(0) = 1^7 = 1$

Thus $(1+x)^7 \approx 1 + 7(x-0)$

or $\boxed{(1+x)^7 \approx 1+7x}$ $(*)$

(b) (4 pts) Use part (a) to approximate 1.02^7 without using a calculator.

$$1.02^7 = (1+0.02)^7 \underset{\substack{\text{by } (*) \\ \text{with } x=0.02}}{\approx} 1 + 7 \times (0.02) = 1.14$$

Thus $\boxed{1.02^7 \approx 1.14}$

9. (20 pts)Given the function $f(x) = \frac{2x^2+1}{x^2-1}$ do the following:

Observe $f(-x) = f(x)$ so the function is even, so ~~that~~ its graph is symmetric w.r.t. y-axis.

(a) Find the domain of $f(x)$.

(b) Carefully compute $f'(x)$ and find the critical point(s). Using a sign chart for the derivative, determine the intervals over which the function is increasing and the intervals over which is decreasing.

(c) Find eventual horizontal and vertical asymptotes. Justify your answer with limits.

(d) Using the results obtained in parts (a), (b), (c), draw the graph of the function indicating the asymptotes, the coordinates and the types of the critical point(s) (the analysis of the second derivative and finding inflection points is **not** required).

(a) $f(x) = \frac{2x^2+1}{x^2-1} = \frac{2x^2+1}{(x-1)(x+1)} \Rightarrow$ Domain: all reals except ± 1

or $x \in (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$

(b) $f'(x) = \frac{4x(x^2-1) - (2x^2+1) \cdot 2x}{(x^2-1)^2} = \frac{2x\left[\cancel{2x^2} - 2 - \cancel{2x^2} - 1\right]}{(x^2-1)^2} = \frac{-6x}{(x^2-1)^2}$

or $f'(x) = \frac{-6x}{(x-1)^2(x+1)^2}$

Sign chart

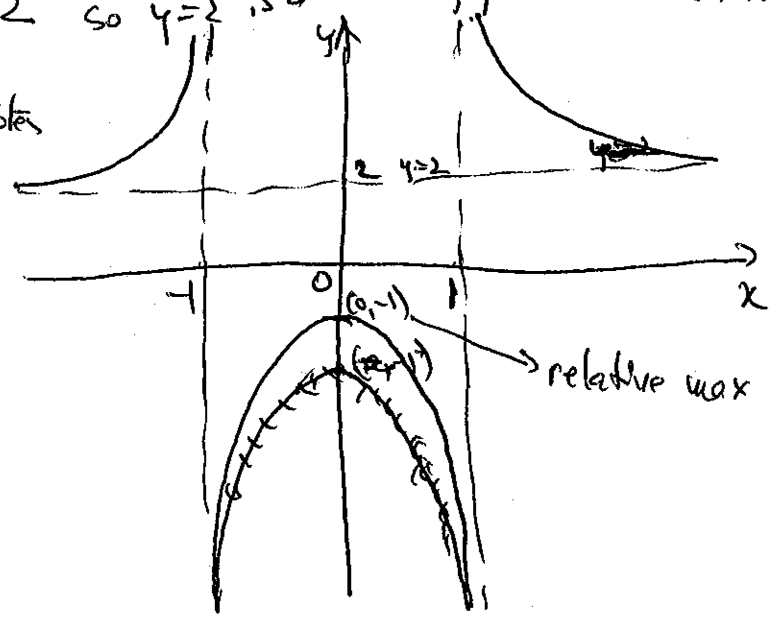
x	$-\infty$	-1	0 (rel. max.)	1	$+\infty$
$f'(x)$	+ + + + + +	‖ + + + + 0	− − −	‖ − − − − −	
$f(x)$	2 ↗ $+\infty$	‖ $-\infty$ ↗ $y=-1$	↘ $-\infty$	‖ $+\infty$ ↘	2

(c) $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \frac{2x^2+1}{x^2-1} = 2$ so $y = 2$ is a horiz asymptote for both $x \to +\infty$, $x \to -\infty$

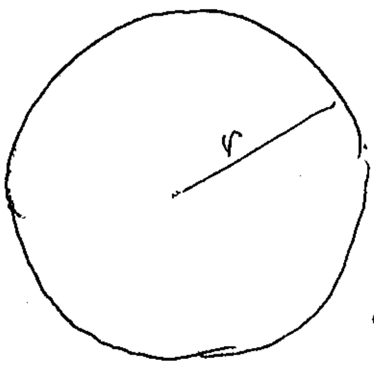
$x = -1$, $x = 1$ are vertical asymptotes

$\lim\limits_{x \to -1^-} \frac{2x^2+1}{x^2-1} = \frac{3}{0^+} = +\infty$

$\lim\limits_{x \to -1^+} \frac{2x^2+1}{x^2-1} = \frac{3}{0^-} = -\infty$



10. (8 pts) Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 2π mi^2/h. At what rate is the radius of the spill increasing when the radius is 2 miles?



$A = \pi r^2$ $A = A(t)$ $r = r(t)$ where t is time in hours

Given $\frac{dA}{dt} = 2\pi$.

We need to find $\frac{dr}{dt} = ?$ when $r = 2$.

Take $\frac{d}{dt}$ of both sides

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} \Rightarrow \frac{dr}{dt} = \frac{\cancel{2\pi}}{\cancel{2\pi}\cdot 2} = \boxed{\frac{1}{2}\ \frac{\text{miles}}{h}}$$

11. (10 pts) Show, with or without Calculus, that for $x > 0$ the function $g(x) = \arctan(x) + \arctan(1/x)$, is a constant. Also, find this constant.

Sol. 1 (with Calculus)

To show $g(x)$ is constant we should show that $g'(x) = 0$ for all $x > 0$.

$$g'(x) = (\arctan x)' + \left(\arctan\left(\tfrac{1}{x}\right)\right)' =$$

$$= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{1+x^2} + \frac{\cancel{x^2}}{1+x^2}\cdot\left(-\frac{1}{\cancel{x^2}}\right) =$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Thus $g'(x) = 0$ for all $x > 0$ $\Rightarrow$ $g(x) = \text{const} = c$

To find c note that $g(1) = \arctan(1) + \arctan(1) = \frac{\pi}{4} + \frac{\pi}{4}$

Thus $c = \frac{\pi}{2}$. You proved the identity $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$

Sol. 2 (without Calculus)

Let $\theta = \arctan x \Rightarrow \tan\theta = x = \frac{x}{1}$

$\varphi = \arctan\left(\frac{1}{x}\right) \Rightarrow \tan\varphi = \frac{1}{x}$

so we can represent θ and φ in the same right angle triangle as complementary angles

Thus $\theta + \varphi = \frac{\pi}{2}$

So $\boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}}$