NAME: Solution Key Panther ID: ____________

Worksheet week 7 - MAC 2311, Spring 2014

1. Many population growth/decay models follow an exponential model. An exponential model is characterized by the property that the rate of change of the population is proportional to its size. Let $P(t) = P_0e^{kt}$ be a certain population at time t, where P_0 and k are parameters.

(a) What is the meaning of P_0? $P(0) = P_0 \cdot e^0 = P_0$

so P_0 is the initial population.

(b) Show that $P(t) = P_0e^{kt}$ satisfies $P'(t) = kP(t)$, so, indeed, the rate of change of the population is proportional to its size, k being the constant of proportionality.

$$P'(t) = (P_0e^{kt})' = P_0(e^{kt})' = P_0e^{kt} \cdot k = k \cdot P(t)$$

(Chain Rule)

2. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 hours.

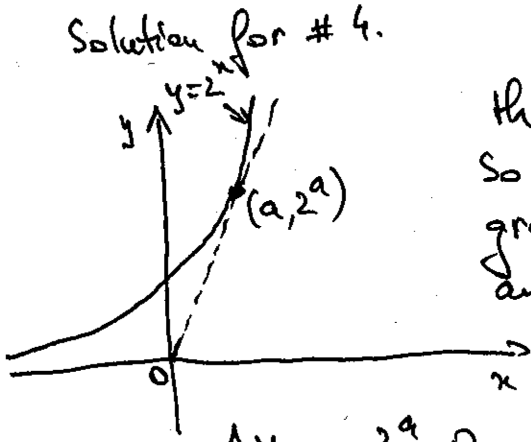
(c) Find the rate of growth after 3 hours.

(d) When will the population reach 10,000?

3. A curve passes through the point $(0, 7)$ and has the property that the slope of the curve at every point P is three times the y-coordinate of P. What is the equation of the curve?

4. Suppose there is a rail-road track along the graph of $f(x) = 2^x$. Let's call it the 2^x-track to infinity! Suppose you live in a city located at the origin $(0, 0)$, so your city is not on the track to infinity. You want to remedy this situation, so you plan to built a railroad from your city to connect smoothly with the 2^x-track to infinity. The problem is that your city can only produce *straight* railroad track. Can you build a straight railroad so that you take a train from your city at $(0, 0)$, you get to some point on $y = 2^x$ and from there the train shifts smoothly onto the 2^x-track to infinity? Be careful, you don't want the train to derail at the connection!

Solution for # 4.



We want our straight rail-road track from the origin to be tangent to $y = 2^x$. So we are looking for a point $(a, 2^a)$ on the graph of $y = 2^x$ so that the line through (0,0) and the point $(a, 2^a)$ is tangent to $y = 2^x$. We express the slope of this line in two different ways:

$$m_{line} = \frac{\Delta y}{\Delta x} = \frac{2^a - 0}{a - 0} = \frac{2^a}{a}$$

$$m_{line} = f'(a) = 2^a \cdot \ln 2$$

$$\Rightarrow \frac{2^a}{a} = 2^a \cdot \ln 2 \Rightarrow a = \frac{1}{\ln 2}$$

Eqn. of the line

$$y - 2^{\frac{1}{\ln 2}} = 2^{\frac{1}{\ln 2}} \ln 2 \left(x - \frac{1}{\ln 2}\right)$$

Solution for Pb. 2. (a) Since the rate of growth of the population is proportional to the population, this must be an exponential function. $P(t) = P_0 \cdot e^{kt}$

$P_0 = 100$ and $P(1) = 420$, so $420 = 100e^{k} \Rightarrow k = \ln(4.2)$

Thus $\boxed{P(t) = 100\, e^{\ln(4.2)t}}$ or $P(t) = 100\left(e^{\ln(4.2)}\right)^{t} = 100 \times (4.2)^{t}$

(b) $P(3) = 100 \times (4.2)^3 = \ldots$ bacteria

(c) $P'(t) = 100 \cdot \ln(4.2)\, e^{\ln(4.2) t}$ so $P'(3) = 100 \ln(4.2) e^{\ln(4.2) \cdot 3}$

$P'(3) = 100 \ln(4.2) \cdot (4.2)^3 \ \frac{\text{bacteria}}{\text{hour}}$

(d) $t = ?$ $P(t) = 10{,}000$

$10{,}000 = 100\, e^{\ln(4.2)t} \Rightarrow 100 = e^{\ln(4.2)t}$ / Apply ln

$\ln 100 = \ln(4.2) \cdot t \Rightarrow t = \frac{\ln(100)}{\ln(4.2)} = \ldots$ hours

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Solution for Pb. 3. At every point the slope is proportional to the function, so again it must be exponential.

$y = f(x) = k \cdot a^{x}$ $\qquad f(0) = 7$ so $k \cdot a^0 = 7 \Rightarrow \underline{k = 7}$

$m_{tan} = f'(x) = k (a^x)' = k a^x \cdot \ln a = (\ln a) \cdot f(x)$

From the statement we have $\ln a = 3$, thus $a = e^3$

Thus, the curve is $\underline{y = f(x) = 7(e^3)^x = 7e^{3x}}$.