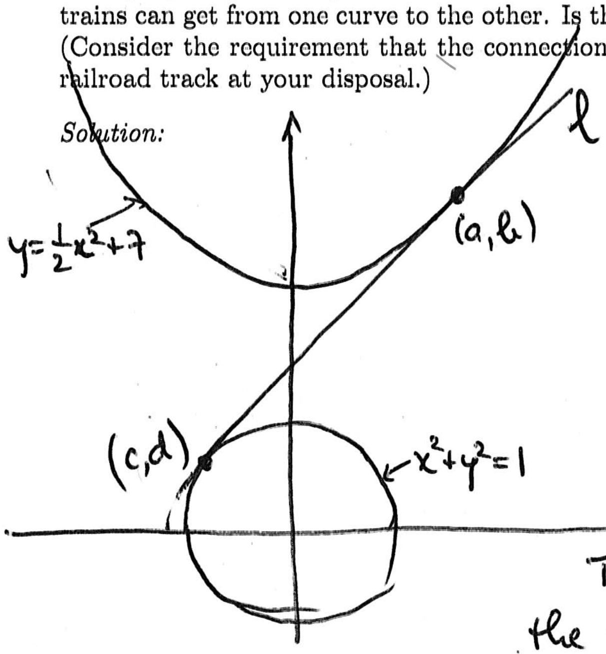
Name: Solution Key Panther ID: ____________

**Mini-Project - MAC2311** Spring 2013 - Instructor: T. Draghici

**Problem 1.** (30 pts) Two railroads have the shapes of the curves $y = \frac{1}{2}x^2 + 7$ and $x^2 + y^2 = 1$. As a chief engineer for a project, you are given the task to build a straight line connection track between the two railroads so that trains can get from one curve to the other. Is this possible? Show the math necessary for the design of this project. (Consider the requirement that the connection be straight to be imposed by the fact that you have only straight railroad track at your disposal.)

*Solution:*




The connection track must be a line tangent to both curves. We want to find the points $(a,b)$, $(c,d)$ which will create a line tangent to both curves.
Since we have 4 unknowns $(a,b,c,d)$, we need to obtain 4 equations relating them and then, hopefully solve the system.
Two equations are obtained by just imposing that the two points are on the two curves.

① $b = \frac{1}{2}a^2 + 7$ ② $c^2 + d^2 = 1$

The slope $m$ of the line $l$ can be given in 3 different ways:

- the derivative of $y = \frac{1}{2}x^2 + 7$ at $x = a$; thus $m = a$
- $\frac{dy}{dx}$ for $x^2+y^2=1$ at $(c,d)$ → we implicit differentiation to get $m = -\frac{c}{d}$
- the slope of the line determined by the two points $m = \frac{d-b}{c-a}$

We obtained thus, two more equations

$a = -\frac{c}{d}$ or ③ $c = -ad$

and $\frac{d-b}{c-a} = -\frac{c}{d}$ or ④ $d^2 - bd = -c^2 + ac$

But notice that ④ can be written as $c^2+d^2 = ac + bd$

which, because of ② becomes $ac + bd = 1$ ④

Thus, we get the 4x4 system below, which is the set up of the problem.

$$\begin{cases} b = \frac{1}{2}a^2 + 7 & (1) \\ c^2 + d^2 = 1 & (2) \\ c = -ad & (3) \\ ac + bd = 1 & (4) \end{cases}$$

Now, if you don't want to do more math by hand, you could input this into wolframalpha (see under "solve system of equations") and you get the 4 possible solutions

| a | b | c | d |
|---|---|---|---|
| $-2\sqrt{2}$ | 11 | $\frac{2\sqrt{2}}{3}$ | $\frac{1}{3}$ |
| $2\sqrt{2}$ | 11 | $-\frac{2\sqrt{2}}{3}$ | $\frac{1}{3}$ |
| $-2\sqrt{6}$ | 19 | $-\frac{2\sqrt{6}}{5}$ | $-\frac{1}{5}$ |
| $2\sqrt{6}$ | 19 | $\frac{2\sqrt{6}}{5}$ | $-\frac{1}{5}$ |

Thus you get 4 possible equations for the connection track which you can write

If you want to get there by hand here is one way to do it:

Substitute c from (3) into (2) to get

$$a^2d^2 + d^2 = 1 \quad \text{or} \quad a^2d^2 = 1 - d^2 \quad \text{or} \quad \boxed{a^2 = \frac{1}{d^2} - 1} \quad (5)$$

Substitute c from (3) and b from (1) into (4) to get

$$-a^2d + \left(\frac{1}{2}a^2 + 7\right)d = 1 \quad \text{or} \quad \boxed{-\frac{1}{2}a^2d + 7d = 1} \quad (6)$$

Substitute $a^2$ from (5) into (6) to get

$$-\frac{1}{2}\left(\frac{1}{d^2} - 1\right)d + 7d = 1, \quad \text{or} \quad -\frac{1}{2d} + \frac{d}{2} + 7d = 1 \quad \text{or}$$

$$15d^2 - 2d - 1 = 0. \text{ The roots are } d_1 = \frac{1}{3}, \; d_2 = -\frac{1}{5}$$

$$(5d+1)(3d-1) = 0$$

Now the 4 solutions follow easily:

For example: If $d_1 = \frac{1}{3}$, by (5) we get $a^2 = 8$ so $a = \pm\sqrt{8} = \pm 2\sqrt{2}$

By (3) $c = \mp \frac{2\sqrt{2}}{3}$ and by (1) $b = 11$

These are the top two solutions in the table

The bottom two solutions are obtained from $d = -\frac{1}{5}$.

Wolframalpha draws the graph as if the lowest points of the graph are on the x-axis. In fact, their horizontal line is not the x-axis, but the line $y=1$. So there is no mistake, but their graph is confusing.

**Problem 2.** (20 pts) The flight of a bee follows the parametric curve $x = t - \cos t$, $y = 3 - 2\sin t$, where $0 \le t \le 4\pi$ is the time in seconds.

(a) At what times is the bee flying horizontally? Find the $(x, y)$ coordinates of the corresponding points.

(b) At what times is the bee flying vertically? Find the $(x, y)$ coordinates of the corresponding points.

(c) Plot this curve in wolframalpha. To my surprise, the plot I got in wolframalpha is not entirely correct. Check if this happens to you. You will receive 5 bonus points if you can point out what is wrong with the graph of this curve in the computer system.

*Solution:* The slope of the bee in flight at a time $t$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\cos t}{1+\sin t}$$

The bee is flying horizontally if the slope is 0.

~~$\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$~~

The bee is flying vertically if the slope is $\pm\infty$.

Note that $\cos t = 0$ when $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ in the interval $[0, 4\pi]$

and $1+\sin t = 0 \Leftrightarrow \sin t = -1$ when $t = \frac{3\pi}{2}, \frac{7\pi}{2}$.

Thus we can say that at $t = \frac{\pi}{2}$ and $t = \frac{5\pi}{2}$ the bee is flying horizontally as $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ at these points.

At $t = \frac{3\pi}{2}$ and $t = \frac{7\pi}{2}$ both $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are 0, so we can't say anything yet.

But
$$\lim_{t\to\left(\frac{3\pi}{2}\right)^-} \frac{-2\cos t}{1+\sin t} = \lim_{t\to\left(\frac{3\pi}{2}\right)^-} \frac{-2\cos t\,(1-\sin t)}{(1+\sin t)(1-\sin t)} = \lim_{t\to\left(\frac{3\pi}{2}\right)^-} \frac{-2\cancel{\cos t}(1-\sin t)}{\cos^{\cancel{2}} t} = +\infty$$

Similarly
$$\lim_{t\to\left(\frac{3\pi}{2}\right)^+} \frac{-2\cos t}{1+\sin t} = -\infty$$

So the bee does fly vertically at the moment $t = \frac{3\pi}{2}$, but it also stops at that moment (as $\frac{dy}{dt} = \frac{dx}{dt} = 0$ at that moment) and changes direction.

The same for $t = \frac{7\pi}{2}$.