Name: Solution Key **Panther ID:** ____________

Exam 1 - MAC2311 - version 1 - Summer B 2018

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
~~*2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.*~~
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) These are True or False questions. No justification required. No partial credit. 2 points each.

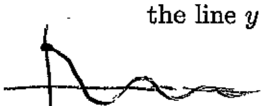
(i) If $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ then $f(x)$ is continuous at $x = 3$. **True** **False**

(ii) By subtraction, $\lim_{x\to +\infty}(x^3 - 1000x^2) = \infty - \infty = 0$. **True** **False**

(iii) For all $a, b > 0$, $\sqrt{a^2+b^2} = a+b$ **True** **False**

(iv) The function $f(x) = \frac{x-2}{\sqrt{x^2+1}}$ is defined and is continuous for all real numbers x. **True** **False**

(v) If $y = L$ is a horizontal asymptote for the function $y = f(x)$, then it is possible for the graph of f to intersect the line $y = L$ infinitely many times. **True** **False**



e.g. $f(x) = \frac{\sin x}{x}$ $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$ (by Squeeze Thm) so $y=0$ is a H.A. but the graph does intersect x-axis infinitely many times (at $x = k\pi$)

(vi) $x = 5$ is a removable discontinuity for the function $f(x) = \dfrac{x^2-25}{x-5}$. **True** **False**

2. (12 pts) An object is dropped from the top of a building. Its position $s(t)$ in feet above the ground t seconds after it was dropped is given by $s(t) = 160 - 16t^2$.

(a) (2 pts) When does the object hit the ground?

$t = ?$ for $s(t) = 0$ $\quad 0 = 160 - 16t^2 \Rightarrow t^2 = 10 \Rightarrow t = \sqrt{10}$ s

(b) (4 pts) Find the average velocity of the object in the time interval $0 \le t \le 2$ seconds.

$$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{s(2)-s(0)}{2-0} = \frac{(\cancel{160} - 16\cdot 2^2) - (\cancel{160 - 0})}{2} = \frac{-16\cdot 4}{2} = -32\ \frac{ft}{s}$$

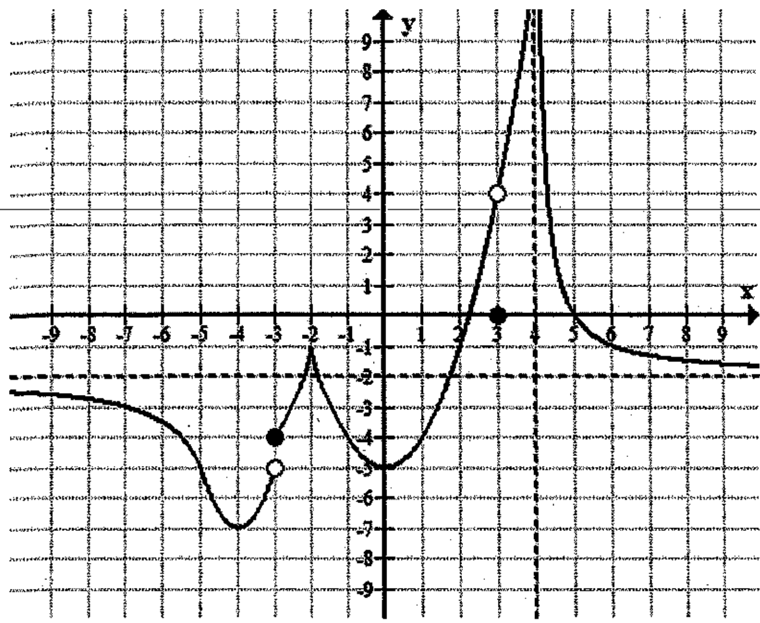
(c) (6 pts) Use limits to find the instantaneous velocity of the object when $t = 2$ seconds.

$$v_{\text{inst at } t=2} = \lim_{h\to 0}\frac{s(2+h)-s(2)}{h} = \lim_{h\to 0}\frac{(160-16(2+h)^2)-(160-16\cdot 2^2)}{h} =$$

$$= \lim_{h\to 0}\frac{\cancel{160} - 16(4+4h+h^2) - \cancel{160} + 16\cdot 4}{h} = \lim_{h\to 0}\frac{\cancel{-64} - 64h - 16h^2 \cancel{+64}}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(-64-16h)}{\cancel{h}_1} = \boxed{-64\ \frac{ft}{s}}$$

Problem 3. (15 pts) The graph of a function f is given below. Answer the questions that follow.



(i) (2 pts) What is the domain of this function? Write your answer in interval form.

$x \in (-\infty, 4) \cup (4, +\infty)$

(ii) (2 pts) What is the range of this function? Write your answer in interval form.

$y \in [-7, +\infty)$

(iii) (5 pts) Find the following limits. Specify if a limit is infinite or does not exist.

a) $\lim_{x\to-\infty} f(x) = -2$ b) $\lim_{x\to-3^-} f(x) = -5$ c) $\lim_{x\to-3} f(x)$ D.N.E. (since $\lim_{x\to-3^+} f(x) = -4 \neq \lim_{x\to-3^-} f(x)$)

d) $\lim_{x\to3} f(x) = 4$ e) $\lim_{x\to4} f(x) = +\infty$

(iv) (4 pts) Is f continuous everywhere? If not, give the x value(s) at which f has a discontinuity. Specify if any of the discontinuities is removable.

No, f is discontinuous at $x=-3$, at $x=3$ and at $x=4$.
From these, only $x=3$ is a removable discontinuity (as $\lim_{x\to3} f(x) = 4$ exists and is finite)

(v) (2 pts) Is f continuous on the interval [-4,0]? Answer and briefly justify.

No, f is discontinuous at $x=-3$.

4. (30 pts) Find the following limits. If the limit is infinite or does not exist, specify so.

(a) (4 pts) $\lim_{x \to 2} \frac{4x - x^3}{x^2 + x - 6}$ $\overset{\frac{0}{0}}{=} \lim_{x\to 2} \frac{x(4-x^2)}{x^2+x-6} = \lim_{x\to 2} \frac{x(2-x)(2+x)}{(x-2)(x+3)} =$

$= \lim_{x\to 2} \frac{(-1)x\cancel{(x-2)}(2+x)}{\cancel{(x-2)}(x+3)} = \boxed{-\frac{8}{5}}$

(b) (4 pts) $\lim_{x \to 9} \frac{\sqrt{x} - 3}{9 - x}$ $\overset{\frac{0}{0}}{=} \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(9-x)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{\cancel{x-9}^{\,1}}{-\cancel{(x-9)}(\sqrt{x}+3)} = -\frac{1}{\sqrt{9}+3} = \boxed{-\frac{1}{6}}$

(c) (4 pts) $\lim_{x \to 2} \frac{x}{x^2 - 4x + 4}$ $= \lim_{x\to 2} \frac{x}{(x-2)^2} = \frac{2}{0^+} = \boxed{+\infty}$

(d) (4 pts) $\lim_{x \to +\infty} \frac{4x^5 + 3x - 2}{4 - 3x^5}$ $\underset{\text{Junk Rule}}{=} \lim_{x\to+\infty} \frac{4x^5}{-3x^5} = \boxed{-\frac{4}{3}}$

(e) (4 pts) $\lim_{x \to +\infty} \tan\left(\frac{\pi x}{4x + 1}\right)$ $\overset{\text{we use here that tan is continuous}}{=} \tan\left(\frac{\pi}{4}\right) = \boxed{1}$

$\lim_{x\to+\infty} \frac{\pi x}{4x+1} \underset{\text{Junk Rule}}{=} \lim_{x\to+\infty} \frac{\pi \cdot x}{4 \cdot x} = \frac{\pi}{4}$

(f) (5pts) $\lim_{x\to 0} \frac{\tan^3(2x)}{x\sin^2(3x)}$ $= \lim_{x\to 0} \frac{\frac{\tan^3(2x)}{(2x)^3}\cdot(2x)^3}{x\,\frac{\sin^2(3x)}{(3x)^2}\cdot(3x)^2} = \lim_{x\to 0} \frac{\left(\frac{\tan(2x)}{2x}\right)^3\cdot 8\cancel{x^3}}{\left(\frac{\sin(3x)}{3x}\right)^2\cdot 9\cancel{x^3}} = \boxed{\frac{8}{9}}$

(g) (5pts) $\lim_{x\to 0} x\sin(1/x)$ $= 0$

Note that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ D.N.E, but the key is that $\sin\left(\frac{1}{x}\right)$ has a bounded oscillation between -1 and 1

So the idea is to apply the squeeze thm.

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$. If $x>0$, then $-x \le x\sin\left(\frac{1}{x}\right) \le x$

If $x<0$, then $-x \ge x\sin\left(\frac{1}{x}\right) \ge x$

In any case $\lim_{x\to 0} x = 0 = \lim_{x\to 0}(-x)$

so $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

5. (9 pts) Given the function $g(x) = \begin{cases} kx^2+2 & \text{if } x\le 0 \\ \frac{\sin(kx)}{x} & \text{if } x>0 \end{cases}$

is there a value of the constant k which will make $g(x)$ continuous everywhere? Justify using limits.

At all other points except $x=0$, the function is continuous, so the question is if there is a value of k making $g(x)$ continuous at $x=0$.

We want $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0)$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(kx^2+2) = k\cdot 0^2 + 2 = \underline{\underline{2}}$. Note also that $g(0) = k\cdot 0^2 + 2 = 2$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{k\sin(kx)}{kx} = k$

Thus, if $\underline{\underline{k=2}}$, $g(x)$ becomes continuous at $x=0$ and hence everywhere

6. (12 pts) Sketch the graph of ONE function $f(x)$ satisfying ALL of the following conditions.

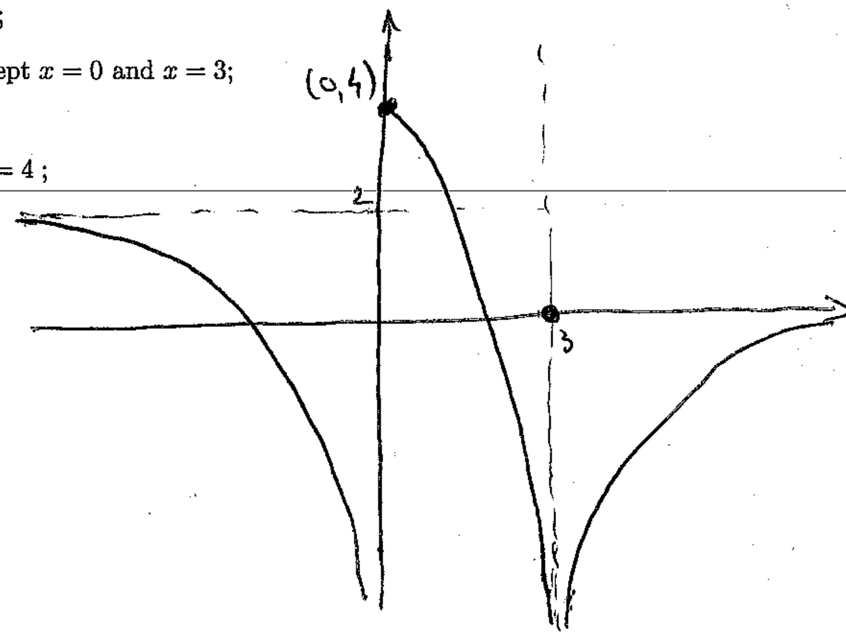
(i) The function is defined for all real numbers;

(ii) The function is continuous everywhere except $x = 0$ and $x = 3$;

(iii) $\lim_{x\to 0^-} f(x) = -\infty$, $f(0) = 4$, $\lim_{x\to 0^+} f(x) = 4$;

(iv) $\lim_{x\to 3} f(x) = -\infty$, $f(3) = 0$, ;

(v) $\lim_{x\to -\infty} f(x) = 2$, $\lim_{x\to +\infty} f(x) = 0$.



7. (10 pts) Use the Intermediate Value Theorem to show that the equation $x^4 = 1 - x^3$ has at least two distinct real solutions and locate each solution in an interval of length 0.5. Justify your work.

$x^4 = 1 - x^3 \iff x^4 + x^3 - 1 = 0$

Let $f(x) = x^4 + x^3 - 1$. It is a polynomial, so it is continuous everywhere.

$f(0) = -1 < 0$, $f(1) = +1 > 0$ $\Rightarrow$ By I.V.T. there is a point $x_1 \in [0,1)$ so that $f(x_1) = 0$

Bisect the interval $f(\frac{1}{2}) = (\frac{1}{2})^4 + (\frac{1}{2})^3 - 1 = \frac{1}{16} + \frac{1}{8} - 1 < 0$. Since $f(1) = 1 > 0$ $\Rightarrow$ I.V.T. guarantees there is a point $\underline{x_1 \in [\frac{1}{2}, 1]}$ so that $f(x_1) = 0$

Next note that $f(-1) = (-1)^4 + (-1)^3 - 1 = -1 < 0$

$f(-2) = (-2)^4 + (-2)^3 - 1 = 16 - 9 > 0$

Bisect again $f(-\frac{3}{2}) = (-\frac{3}{2})^4 + (-\frac{3}{2})^3 - 1 = \frac{81}{16} - \frac{27}{8} - \frac{1}{1} = \frac{81 - 54 - 16}{16} > 0$

So $f(-1) < 0$ and $f(-\frac{3}{2}) > 0$, thus by I.V.T. there is a point $x_2 \in [-\frac{3}{2}, -1]$ so that $f(x_2) = 0$

Thus, the equation has at least two real roots, one in the interval $[\frac{1}{2}, 1]$, one in the interval $[-\frac{3}{2}, -1]$.

8. (10 pts) Choose ONE of the following. Only ONE will receive credit.

(A) State and prove the quadratic formula.

(B) Prove that $\sin x \leq x \leq \tan x$ for any angle $x \in [0, \pi/2)$.

see text or notes