Solutions to Test #1

1a) True. Because $f(x)$ is continuous at 3, we have $4 = \lim_{x\to 3} f(x) = f(3)$.
1b) False. Square roots are always positive, so the limit is $+\infty$.
1c) False. Only $\lim_{x\to 0}\sin(2x)/x = 2$, but it is certainly not true that $\sin(2x)/x = 2$ for all numbers x.
1d) False. Because $\sec(x) = 1/\cos(x)$ and $\cos(x) = 0$ for $x = \pi/2 + n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$, $\sec(x)$ is not defined and thus not continuous at $x = \pi/2 + n\pi$.
1e) False. Consider the example $\lim_{x\to 0} x/x = 1$.
1f) True.

2a) The object hits the ground when $s(t) = 0$. We solve $0 = 160 - 16t^2 = 16(10 - t^2)$ and get $t = \pm\sqrt{10}$. Thus take $t = \sqrt{10}$.

2b) The average velocity over the interval $[0, 3]$ is:

$$\frac{s(3) - s(0)}{3 - 0} = \frac{160 - 16(3)^3 - 160}{3} = -48 \text{ ft./s..}$$

2c) The instantaneous velocity at time $t = 3$ is

$$\begin{aligned}
\lim_{h\to 0}\frac{s(3+h) - s(3)}{h} &= \lim_{h\to 0}\frac{160 - 16(3+h)^2 - 160 + 16(3)^2}{h} \\
&= \lim_{h\to 0}\frac{-16(3^2 + 6h + h^2) + 16(3)^2}{h} \\
&= \lim_{h\to 0}\frac{-16(3^2 + 6h + h^2) + 16(3)^2}{h} = \lim_{h\to 0}\frac{-16(6h + h^2)}{h} \\
&= \lim_{h\to 0}\frac{-16h(6 + h^2)}{h} \\
&= \lim_{h\to 0} -16(6 + h^) = -16(6) = -96 \text{ ft./s..}
\end{aligned}$$

3) The function $g(x)$ is clearly continuous everywhere except perhaps at $x = 2$. We compute:

$$\begin{aligned}
\lim_{x\to 2^-} g(x) &= \lim_{x\to 2^-} kx^2 - 1 = \lim_{x\to 2} kx^2 - 1 = 4k - 1. \\
\lim_{x\to 2^+} g(x) &= \lim_{x\to 2^+} 2x + 3 = \lim_{x\to 2} 2x + 3 = 7.
\end{aligned}$$

Because $\lim_{x\to 2} g(x)$ exists if and only if $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x)$, we see that $\lim_{x\to 2} g(x)$ exists if and only if $4k - 1 = 7$ or $k = 2$. If $k = 2$, then

$$\lim_{x\to 2} g(x) = \lim_{x\to 2+} g(x) = \lim_{x\to 2} g(x) = 7.$$

If $k = 2$, we also have $g(2) = 2(2)^2 - 1 = 7$. Hence, if $k = 2$, we have $\lim_{x\to 2} g(x) = 7 = g(2)$ so $g(x)$ is continuous at 2 and hence everywhere.

4a) $\lim_{x\to -2} \frac{x^2-2x-8}{x^2-4} = \lim_{x\to -2} \frac{(x-4)(x+2)}{(x-2)(x+2)} = \lim_{x\to -2} \frac{(x-4)}{(x-2)} = \frac{-6}{-4} = \frac{3}{2}$.

4b) $\lim_{x\to -2} \frac{x^2-2}{x^2+4} = \frac{0}{(-2)^2+4} = 0$

4c) $\lim_{x\to 3} \frac{|x-3|}{x^2-6x+9} = \lim_{x\to 3} \frac{|x-3|}{(x-3)^2} = \lim_{x\to 3} \frac{1}{|x-3|} = +\infty$.
Note that $|x-3| \geq 0$ and $(x-3)^2 = |x-3|^2 \geq 0$, so the quotient must also be ≥ 0.

4d) $\lim_{x\to -2^+} \frac{x-1}{x+2} = -\infty$. Observe that for $x - 2$, we have $x+2 > 0$ but for $-2 < x < -1$, $x - 1 < 0$. Hence the quotient $(x-1)/(x+2) < 0$ for $x \in (-2, -1)$ so the limit is $-\infty$.

4e) $\lim_{x\to -\infty} \frac{4x^5+3x-2}{3x^5+4} = \lim_{x\to -\infty} \frac{4x^5}{3x^5} = \frac{4}{3}$

4f) We use the Squeeze theorem. Observe that for all x, $-1 \leq \sin(x^2) \leq 1$. Hence,

$$-\frac{1}{x^2} \leq \frac{\sin(x^2)}{x^2} \leq \frac{1}{x^2}.$$

We know that $\lim\limits_{x\to\infty} \pm 1/x^2 = 0$, so, by the Squeeze theorem $\lim\limits_{x\to+\infty} \frac{\sin(x^2)}{x^2} = 0$.

4g)

$$\lim_{x\to 0} \frac{\tan(5x)}{x+\sin x} = \lim_{x\to 0} \frac{\frac{\tan(5x)}{5x}\cdot(5x)}{x+\frac{\sin x}{x}\cdot x} == \lim_{x\to 0} \frac{\frac{\tan(5x)}{5x}\cdot(5x)}{x(1+\frac{\sin x}{x})}$$
$$= \lim_{x\to 0} \frac{\frac{\tan(5x)}{5x}\cdot 5}{(1+\frac{\sin x}{x})} = \frac{1\cdot 5}{1+1} = \frac{5}{2}.$$

4h)

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} &= \lim_{x\to 0} \frac{1-\cos(3x)}{x^2}\frac{1+\cos(3x)}{1+\cos(3x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2(3x)}{x^2}\frac{1}{1+\cos(3x)} \\
&= \lim_{x\to 0} \frac{\sin^2(3x)}{x^2}\frac{1}{1+\cos(3x)} \\
&= \lim_{x\to 0} 3\frac{\sin(3x)}{3x}3\frac{\sin(3x)}{3x}\frac{1}{1+\cos(3x)} \\
&= (3)(3)\frac{1}{1+1} = \frac{9}{2}.
\end{aligned}$$

5) The function $f(x) = x^4 + x - 1$ is a polynomial so it is continuous everywhere. Thus we may apply the IVT on any interval. We are trying to solve the equation $f(x) = 0$. Because $-1 = f(0) < 0 < 1 = f(1)$, the IVT implies that there is a solution in $[0, 1]$. We compute $f(1/2) = (1/16) + (1/2) - 1 = (1 + 8 - 16)/16 < 0$. Thus the IVT implies that there is a solution in the interval $[1/2, 1]$. This gives one solution to the desired precision. Next we compute $f(-2) = 13 > 0 > -1 = f(-1)$ so there is a solution in $[-2, -1]$. We compute:

$$f(-3/2) = (81/16) - (3/2) - 1 = \frac{81 - 24 - 16}{16} > 0,$$

so we have the inequalities $f(-3/2) > 0 > f(-1)$ and the IVT implies that there is a solution in $[-3/2, -1]$.

6) There are many possible sketches satisfying these requirements.

7a) The definition can be read in the textbook.

7b) We compute $|f(x) - L| = |(5x - 7) - 8| = 5|x - 3|$. Thus,

$$\text{if } |x - 3| < \delta, \text{ then } |f(x) - L| = 5|x - 3| < 5\delta.$$

Hence, given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{5}$. With this choice, if $0 < |x - 3| < \delta = \varepsilon/5$, we have $|f(x) - L| = 5|x - 3| < 5\delta = 5\frac{\varepsilon}{5} = \varepsilon$ and this proves the desired limit.

7c) Compute

$$|f(x) - L| = |(2x^2 + 1) - 19| = |2x^2 - 18| = 2|x - 3||x + 3|.$$

We need to control the size of the factor $|x + 3|$.

Make a preliminary choice, $\delta \le 1$.

Then $|x - 3| < \delta \le 1$ implies $|x - 3| < 1$, so $-1 < x - 3 < 1$. By adding 6 to all sides, we get $5 < x + 3 < 7$. Hence, we've shown that

$$\text{if } |x - 3| < \delta \le 1, \text{ then } |x + 3| < 7, \text{ hence, further,}$$

$$\text{If } |x - 3| < \delta \le 1, \text{ then } |f(x) - L| = 2|x - 3||x + 3| < 2\delta \cdot 7 = 14\delta, \tag{0.1}$$

So, given $\varepsilon > 0$, make the final choice, $\delta = \min(1, \varepsilon/14)$. With this choice $\delta \le 1$ and $\delta \le \varepsilon/14$, so we can complete inequality (0.1) to:

$$\text{if } |x - 3| < \delta = \min(1, \varepsilon/14), \text{ then } |f(x) - L| = 2|x - 3||x + 3| < 2\delta \cdot 7 = 14\delta \le 14\frac{\varepsilon}{14} = \varepsilon,$$

as required.