Solutions to Exam #2

1a) True. This follows from the power law and the differentiation rules $(cf)' = cf'$ and $(f+g)' = f' + g'$.

1b) False. You can't evaluate the function (i.e. substitute in 2 for x) before differentiating.

1c) True. We have $y' = \cos(x)$ and $y'' = -\sin(x) = -y$.

1d) False. We compute $(\cos(g(x)))'$ by using the chain rule, not the product rule. The correct expression would be $h'(x) = -\sin(g(x))g'(x)$.

1e) True. Just apply the chain rule.

1f) True. We compute $y' = 1/x$ so the slope of the tangent line at $(a, \ln(a))$ is $1/a$ and $\lim_{a\to 0^+} 1/a = \infty$.

2a) $\frac{d}{dx}\left(3x^5 - 2\sqrt{x} + 10^x\right) = 3(5)x^4 - 2\frac{1}{2\sqrt{x}} + \ln(10)10^x$

2b)

$$\frac{d}{dx}\left(\frac{\arcsin x}{x^2+4}\right) = \frac{\frac{1}{\sqrt{1-x^2}}(x^2+4) - 2x\arcsin(x)}{(x^2+4)^2}$$

2c)

$$\frac{d}{dx}\left(e^{\cos x}\tan x\right) = e^{\cos(x)}(-\sin(x))\tan(x) + e^{\cos(x)}\sec^2(x).$$

2d)

$$\begin{aligned}\frac{d}{dx}\left(\ln(\sec(\arctan x))\right) &= \frac{\sec(\arctan(x))\tan(\arctan(x))\frac{1}{1+x^2}}{\sec(\arctan(x))} \\ &= \tan(\arctan(x))\frac{1}{1+x^2} \\ &= \frac{x}{1+x^2}\end{aligned}$$

2e) Take the logarithm of each side and simplify:

$$\ln(y) = \ln\left((1+x^2)^{1/x}\right) = \frac{1}{x}\ln(1+x^2)$$

We then differentiate both sides:

$$\begin{aligned}\frac{d}{dx}\ln(y) &= \frac{d}{dx}\left(\frac{1}{x}\ln(1+x^2)\right)\\ \frac{y'}{y} &= \left(-\frac{1}{x^2}\right)\ln(1+x^2) + \left(\frac{1}{x}\right)\frac{2x}{1+x^2}\end{aligned}$$

Hence,

$$\begin{aligned}y' &= y\left(\left(-\frac{1}{x^2}\right)\ln(1+x^2) + \left(\frac{1}{x}\right)\frac{2x}{1+x^2}\right)\\ &= (1+x^2)^{1/x}\left(-\frac{\ln(1+x^2)}{x^2} + \frac{2}{1+x^2}\right)\end{aligned}$$

3) Let y be the altitude of the rocket above the launch pad (in kilometers). Let z be the distance from the rocket to the radar station. You should make a picture and mark these variables on your picture. Note that both y and z vary with time, whereas the horizontal distance between the launch pad and radar station is a constant (30 km). From Pythagorean theorem, $z^2 = y^2 + (30)^2$. Differentiate both sides of this equality with respect to t:

$$2z\frac{dz}{dt} = 2y\frac{dy}{dt}.$$

When $z = 50$, We have $(50)^2 = y^2 + (30)^2$ so $y = 40$. Thus, if $z = 50$ and $dz/dt = 60$, we have

$$2(50)(60) = 2(40)\frac{dy}{dt},$$

so $\frac{dy}{dt} = \frac{2(50)(60)}{2(40)} = \frac{3000}{40} = 75$ kilometers per minute.

4a) We compute $f'(x) = \frac{1}{4}x^{-3/4}$. Then $f(x_0) = f(1) = (1)^{1/4} = 1$ and $f'(x_0) = f'(1) = \frac{1}{4}(1)^{-3/4} = \frac{1}{4}$. Hence, the linear approximation is

$$x^{1/4} \approx 1 + \frac{1}{4}(x-1).$$

4b) Using the formula

$$x^{1/4} \approx 1 + \frac{1}{4}(x-1),$$

we see that

$$(.92)^{1/4} \approx 1 + \frac{1}{4}(0.92-1) = 1 + \frac{-.08}{4} = 0.98.$$

5) We differentiate implicitly:

$$4(x^2+y^2)(2x+2yy') = 25(2x-2yy').$$

Now substitute in $x = 3$ and $y = 1$ into the preceding equation and get:

$$\begin{aligned}
4(3^2 + 1^2)(2(3) + 2(1)y') &= 25(2(3) - 2(1)y') \\
4(10)(6 + 2y') &= 25(6 - 2y') \\
240 + 80y' &= 150 - 50y' \\
130y' &= 150 - 240 = -90 \\
y' &= -\frac{90}{130} = -\frac{9}{13}
\end{aligned}$$

Thus, the slope of the line tangent to the curve at $(3, 1)$ is $-9/13$ so the equation of the line is:

$$y - 1 = -\frac{9}{13}(x - 3).$$

6a) We compute:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \lim_{h \to 0} \frac{\cos(x + h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h \to 0} \sin(x)\frac{\sin(h)}{h} \\
&= \cos(x)\lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x)\lim_{h \to 0} \frac{\sin(h)}{h} \\
&= \cos(x)(0) - \sin(x)(1) \\
&= -\sin(x).
\end{aligned}$$

6b) We differentiate both sides of the identity $\cos(\arccos(x)) = x$ and get

$$\begin{aligned}
\frac{d}{dx}\left(\cos(\arccos(x))\right) &= \frac{d}{dx}x \\
-\sin(\arccos(x))\frac{d}{dx}\arccos(x) &= 1
\end{aligned}$$

Thus,

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sin(\arccos(x))}.$$

We simplify $\sin(\arccos(x))$ by drawing a right triangle containing the angle $\theta = \arccos(x)$. If the side adjacent to the angle θ has length x, then the hypotenuse must have length 1. The side opposite the angle θ must then have length $\sqrt{1 - x^2}$. Hence, $\sin(\arccos(x)) = \sin(\theta) = \sqrt{1 - x^2}/1$ and we have

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sin(\arccos(x))} = -\frac{1}{\sqrt{1 - x^2}}.$$

7) The chain rule tells us that

$$h'(x) = f'(g(x))g'(x).$$

We differentiate again and get:

$$\begin{aligned} h''(x) &= \frac{d}{dx}\big(f'(g(x))g'(x)\big) \\ &= \left(\frac{d}{dx}\big(f'(g(x))\big)\right)g'(x) + f'(g(x))\frac{d}{dx}g'(x) \quad \text{by the product rule} \\ &= f''(g(x))g'(x)g'(x) + f'(g(x))g''(x). \end{aligned}$$