Solutions to Test #3

1a) False. An inflection point occurs where f'' changes sign which it cannot do at $x = 5$ if $f''(5) \neq 0$.

1b) True. The function is strictly increasing, so it does not have an absolute maximum on $(-\infty, \infty)$.

1c) False. Consider $\int 1 \cdot 1\ dx = \int 1\ dx = x + C$ while

$$\left(\int 1\ dx\right)\left(\int 1\ dx\right) = (x + C)(x + D) = x^2 + Ex + F.$$

1d) True. Any continuous function on a closed, finite interval has an absolute minimum on that interval (the extreme value theorem).

1e) False. You need, in addition, that the function f be differentiable on $(0, 4)$ to apply Rolle's Theorem (or MVT).

1f) True because $y = 3$ is a horizontal asymptote.

2a)

$$\int \left(3\cos x + \frac{1}{\sqrt{1 - x^2}}\right) dx = 3\sin(x) + \arcsin(x) + C$$

2b)

$$\int \frac{x^2 - 6}{\sqrt{x}}\ dx = \int x^{3/2} - 6x^{-1/2}\ dx = \frac{2}{5}x^{5/2} - 6(2)x^{1/2} + C.$$

2c) Make the substitution $u = \tan(x)$ so $du = sec^2(x)\ dx$ and the integral becomes

$$\int \tan(x)\sec^2(x)\ dx = \int u\ du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2(x) + C$$

2d) Make the substitution $u = -3x^2$ do $du = -6x\ dx$ or $-(1/6)\ du = x\ dx$ and the integral becomes

$$\int xe^{-3x^2}\ dx = \int -\frac{1}{6}e^u\ du = -\frac{1}{6}e^u + C = -\frac{1}{6}e^{-3x^2} + C$$

2e) Make the substitution $u = x^2$ so $du = 2x\ dx$ or $(1/2)\ du = x\ dx$ and the integral becomes:

$$\int \frac{x}{x^4 + 1}\ dx = \int \frac{1}{2}\frac{1}{u^2 + 1}\ du = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x^2) + C$$

3) We compute $f'(x) = 3x^2 - 6x + 1$. Hence $f(x)$ is differentiable on all of $[-2, 3]$ and critical points for $f(x)$ only occur where $0 = f'(x) = 3x^2 - 6x = 3x(x-2)$, i.e. at $x = 0, 2$. We compare the value of $f(x)$ at these critical points and at the end points:

$$\begin{aligned} f(-2) &= (-2)^3 - 3(-2)^2 + 1 = -8 - 12 + 1 = -19, \\ f(0) &= 0^3 - 3(0)^2 + 1 = 1, \\ f(2) &= 2^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3, \\ f(3) &= 3^3 - 3(3)^2 + 1 = 1. \end{aligned}$$

Hence, the absolute minimum is $f(-2) = -19$ while the absolute maximum is $f(0) = f(3) = 1$.

4a) Because

$$f(x) = \frac{1}{x^2 - 4x + 3} = \frac{1}{(x-3)(x-1)},$$

we see that $f(x)$ has vertical asymptotes at $x = 1$ and $x = 3$. By the garbage rule,

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - 4x + 3} = \lim_{x \to \infty} \frac{1}{x^2} = 0,$$

so $f(x)$ has a horizontal asymptote at $y = 0$.

4b) We compute:

$$f'(x) = \frac{-(2x-4)}{(x^2 - 4x + 3)^2} = \frac{-(2x-4)}{((x-3)^2(x-1)^2}$$

Because $(x-3)^2(x-1)^2 \geq 0$, we see that the sign of $f'(x)$ is the same as that of $-(2x-4)$ which is positive on $(-\infty, 2)$ and negative on $(2, \infty)$. Noting that $f'(x)$ is not defined at $x = 1$ and at $x = 3$, we can then say that $f'(x) > 0$ for $x \in (-\infty, 1)$ and for $x \in (1, 2)$ while $f'(x) < 0$ for $x \in (2, 3)$ and $x \in (3, \infty)$. Hence $f(x)$ is increasing on $(-\infty, 1)$ and on $(1, 2)$ while $f(x)$ is decreasing on $2, 3)$ and on $(3, \infty)$.

4c) Sketch the graph.

5a) There are no vertical asymptotes as $f(x)$ is defined for all $x \in (-\infty, \infty)$ and $\lim_{x\to\pm\infty} ln(1+x^2) = \infty$ so there are no horizontal asymptotes.

5b) We compute:

$$f'(x) = \frac{2x}{1+x^2}.$$

Because $1 + x^2 > 0$ for all x, the sign of $f'(x)$ equals that of $2x$. Hence, $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

5c) We compute

$$f''(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2} = \frac{2(1-x)(1+x)}{(1+x^2)^2}.$$

Because $(1 + x^2)^2 > 0$ for all x, the sign of f'' equals that of $2(1 - x)(1 + x)$. Hence, $f''(x) < 0$ for $x \in (-\infty, 1)$ and for $x \in (1, \infty)$ while $f''(x) > 0$ for $x \in (-1, 1)$. Thus $f(x)$ is concave down on $(-\infty, 1)$ and on $(1, \infty)$ while it is concave up on $(-\infty, 1)$. Observe that there are inflection points at $x = \pm 1$.

5d) Sketch the graph.

6a) Using L'Hopital's rule, we compute

$$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2} \overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{2\sin(2x)}{2x} = \lim_{x\to 0} \frac{\sin(2x)}{x} \overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{2\cos(2x)}{1} = \frac{2(1)}{1} = 2.$$

6b) We rewrite the limit as:

$$\lim_{x\to 0} \Big(1+5x\Big)^{\frac{2}{x}} = \lim_{x\to 0} \exp\left(\frac{2}{x}\ln(1+5x)\right).$$

We compute

$$\lim_{x\to 0} \frac{2\ln(1+5x)}{x} \overset{\text{L.H.}}{=} \lim_{x\to 0} \frac{\frac{2}{1+5x}\cdot 5}{1} = \lim_{x\to 0} \frac{10}{1+5(0)} = 10.$$

Hence, the limit equals,

$$\lim_{x\to 0} \exp\left(\frac{2}{x}\ln(1+5x)\right) = \exp\left(lim_{x\to 0}\frac{2}{x}\ln(1+5x)\right) = e^{10}.$$

7) Let ℓ, w be the length and width of the top and bottom of the box and let h be the height. We want to find the maximum value for the volume,

$$V = \ell wh.$$

We are told $\ell = 2w$. The surface area of the box is

$$1500 = 2\ell w + 2\ell h + 2wh.$$

Substituting in $\ell = 2w$, this gives us the relation between w and h,

$$1500 = 4w^2 + 4wh + 2wh = 4w^2 + 6wh.$$

We solve this for h in terms of w and get

$$h = \frac{1500 - 4w^2}{6w}.$$

Then, we can write the volume as a function of w by:

$$V = \ell wh = 2w^2 h = 2w^2\frac{1500 - 4w^2}{6w} = \frac{1}{3}\left(1500w - 4w^3\right).$$

The constraints $w > 0$ and $h = (1500 - 4w^2)/(6w) > 0$ imply that $1500 > 4w^2$ or $w < \sqrt{1500/4} = \sqrt{375} = 5\sqrt{15}$. Hence, we are trying to find the absolute maximum of $V(w)$ on the interval $(0, 5\sqrt{15})$. We compute:

$$V'(w) = \frac{1}{3}\left(1500 - 12w^2\right) = 500 - 4w^2.$$

Thus $V'(w) = 0$ implies $500 = 4w^2$ or $w = \pm 125 = \pm 5\sqrt{5}$. Since $V''(w) = -8w < 0$, the function $V(w)$ is always concave down, so the critical point $w = 5\sqrt{5}$ must be an absolute maximum. Hence, the desired dimensions are:

$$w = 5\sqrt{5},\ \ell = 2w = 10\sqrt{5},\ h = \frac{1500 - 4w^2}{6w} = \frac{1500 - 4(125)}{30\sqrt{5}} = \frac{1000}{30\sqrt{5}}.$$

8a) Because $a = v'(t)$, we have

$$v(t) = \int a\ dt = at + C_1.$$

We use the initial condition $v_0 = v(0)$ to compute $v_0 = a(0) + C_1$ and get $C_1 = v_0$. Thus

$$v(t) = at + v_0.$$

Because $at + v_0 = s'(t)$, we have

$$s(t) = \int at + v_0\ dt = \frac{1}{2}at^2 + v_0 t + C_2.$$

We use the initial condition $s_0 = s(0)$ to compute $s_0 = s(0) = (1/2)a(0)^2 + v_0(0) + C_2$ and get $C_2 = s_0$. Thus,

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0.$$

8b) Because acceleration is constant, we have $a = -32$ and by (8a), we can write

$$s(t) = -16t^2 + v_0 t + s_0.$$

We set $s_0 = s(0) = 0$ because the arrow is initially on the ground. Using the given information that $s(3) = 120$, we solve for v_0:

$$120 = s(3) = -16(3)^2 + v_0(3),$$

and get $3v_0 = 120 + 16(9) = 264$ so $v_0 = 264/3 = 88$ (in feet per second).