Mean Value Theorem Worksheet:

1) Consider the function $f(x) = x(x-2)^2$ on the interval $[0, 2]$.

a) Compute $f(0)$ and $f(2)$.

b) Does Rolle's theorem apply to $f(x)$ on the interval $[0, 2]$?

c) Find the point(s) $c \in (0, 2)$ whose existence is guaranteed by Rolle's theorem.

2) Consider the function $f(x) = x - \frac{1}{x}$ on the interval $[1, 3]$. Find the point(s) in $(1, 3)$ that are guaranteed to exist by the Mean Value Theorem. Sketch a picture of the graph of $f(x)$ on $[1, 3]$ including at least one such point and illustrating the Mean Value Theorem.

3) Consider the function $f(x) = |4x - 8|$ on the interval $[1, 3]$.
a) Compute $f(1)$ and $f(3)$.
b) Is there $c \in (1, 3)$ such that $f'(c) = 0$?
c) Why does Rolle's Theorem not apply to $f(x)$ on $[1, 3]$?
d) Does the Mean Value theorem apply to $f(x)$ on the interval $[3, 5]$? If so find the point(s) $c \in [3, 5]$ whose existence is guaranteed by the MVT.

4) Suppose that a state police force has deployed an automated radar tracking system on a highway that has a speed limit of 65 mph. A driver passes through one radar detector at 1pm and is traveling 60 mph at that moment. Then, the driver passes through a second radar detector 60 miles away at 1:45pm, again traveling 60 mph at that moment. However, a speeding ticket is being issued for this driver. Argue with Calculus that the speeding ticket is justified.

5) It is intuitively obvious that if $f'(x) > 0$ for all $x \in (a, b)$ then $f(x)$ is increasing. However, the more one thinks about it, this assertion becomes more troubling. The value of $f'(x)$ only controls the slope of the tangent line at the point x: why should it say anything about values of $f(x')$ when x' is near x? In these problems, we discuss how the Mean Value Theorem gives a rigorous proof of this assertion.

To give such a proof, we need a precise definition of the word *increasing* when it describes a function on an interval. Here is such a definition. We say that a function $f(x)$ is *increasing* on (a, b) if for all $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, $f(x_1) < f(x_2)$.

a) If $x_1 < x_2$ what can you say about the sign of $x_2 - x_1$?

b) If $x_1 < x_2$ and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0,$$

what can you say about the sign of $f(x_2) - f(x_1)$?

c) In (a) and (b), we have shown that if

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

then $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Explain how the Mean Value Theorem tells us that if $f'(x) > 0$ for all $x \in (a, b)$ then $f(x)$ is increasing on (a, b).