NAME: Solution Key   Panther ID: ____________

Exam 2 - MAC 2311   Spring 2016

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (36 pts) Find dy/dx (6 pts each). You do not have to simplify, except in part (c).

(a) $y = 4\arcsin x + \frac{2}{\sqrt{x}} + \pi^2$ $= 4\arcsin x + 2x^{-\frac{1}{2}} + \underbrace{\pi^2}_{\text{constant}}$

$$y' = \frac{4}{\sqrt{1-x^2}} + \cancel{2}\cdot\left(-\frac{1}{\cancel{2}}\right)x^{-\frac{3}{2}} + 0$$

or $$y' = \frac{4}{\sqrt{1-x^2}} - \frac{1}{x\sqrt{x}}$$

(b) $y = \frac{x^3 + \sqrt[3]{x}}{3^x}$ $= \frac{x^3 + x^{\frac{1}{3}}}{3^x}$

$$y' = \left(\frac{x^3 + x^{\frac{1}{3}}}{3^x}\right)' \underset{\substack{\uparrow \\ \text{quotient rule}}}{=} \frac{\left(3x^2 + \frac{1}{3}x^{-\frac{2}{3}}\right)\cdot 3^x - \left(x^3 + x^{\frac{1}{3}}\right)\cdot 3^x \ln 3}{(3^x)^2}$$

(c) $y = \arctan(e^x) + \arctan(e^{-x})$ **Note:** 2 bonus points if you fully simplify your answer here.

$$y' = (\arctan(e^x))' + (\arctan(e^{-x}))' = \frac{1}{1+(e^x)^2}\cdot(e^x)' + \frac{1}{1+(e^{-x})^2}(e^{-x})'$$

$$y' = \frac{e^x}{1+e^{2x}} - \frac{e^{-x}}{1+e^{-2x}} = \frac{e^x}{1+e^{2x}} - \frac{\frac{1}{e^x}}{1+\frac{1}{e^{2x}}} = \frac{e^x}{1+e^{2x}} - \frac{\frac{1}{e^x}}{\frac{e^{2x}+1}{e^{2x}}}$$

so $$y' = \frac{e^x}{1+e^{2x}} - \frac{1}{\cancel{e^x}}\cdot\frac{\cancel{e^{2x}}e^x}{e^{2x}+1} \quad \text{so} \quad \boxed{y' = 0}\,!!$$

There must be another reason for this result. Can you figure this out?

(d) $y = \sec^2(x^3) = \left(\sec(x^3)\right)^2$

$$y' = \left(\left(\sec(x^3)\right)^2\right)' = 2\sec(x^3)\cdot\left(\sec(x^3)\right)' =$$

$$= 2\sec(x^3)\cdot\sec(x^3)\tan(x^3)\cdot(x^3)' = 2\sec(x^3)\sec(x^3)\tan(x^3)\cdot 3x^2 =$$

$$= 6x^2\sec^2(x^3)\tan(x^3)$$

(e) $y = \sqrt{1+(\tan x)^{-2}}$

Shortest solution: Notice that $(\tan x)^{-2} = \frac{1}{(\tan x)^2} = (\cot x)^2 = \cot^2 x$

and use the identity $1+\cot^2 x = \csc^2 x$

Thus $y = \sqrt{1+(\tan x)^{-2}} = \sqrt{1+(\cot x)^2} = \sqrt{\csc^2 x} \underset{\substack{\uparrow \\ \text{assume } \csc x > 0}}{=} \csc x$

Then $y' = (\csc x)' = -\csc x \cot x$

(f) $y = (x^2+1)^{1/x}$ $\leftarrow$ Needs logarithmic differentiation

$$\ln y = \ln\left((x^2+1)^{\frac{1}{x}}\right) = \frac{1}{x}\ln(x^2+1) = \frac{\ln(x^2+1)}{x} \quad \left(\text{Apply } \frac{d}{dx}\right.$$

$$(\ln y)' = \left(\frac{\ln(x^2+1)}{x}\right)'$$

$$\frac{1}{y}\cdot y' = \frac{\frac{1}{x^2+1}\cdot 2x\cdot x - \ln(x^2+1)\cdot 1}{x^2} = \frac{\frac{2x^2}{x^2+1} - \ln(x^2+1)}{x^2}$$

Thus $y' = y\cdot\dfrac{\frac{2x^2}{x^2+1} - \ln(x^2+1)}{x^2} = \boxed{(x^2+1)^{\frac{1}{x}}\cdot\dfrac{\frac{2x^2}{x^2+1} - \ln(x^2+1)}{x^2}}$

2. (16 points) These are True or False questions. Circle your answer AND give a brief justification.

(a) If $y = \cos(f(x))$, then $dy/dx = -\sin(f'(x))$. **True** **False**

Justification: By Chain Rule, $\frac{dy}{dx} = -\sin(f(x)) \cdot f'(x)$

(b) If $g(x) = f(x)\sin x$ then $g'(0) = f(0)$. **True** **False**

Justification: By product rule, $g'(x) = f'(x)\sin x + f(x)\cos x$

so $g'(0) = f'(0)\cdot 0 + f(0)\cdot 1 = f(0)$

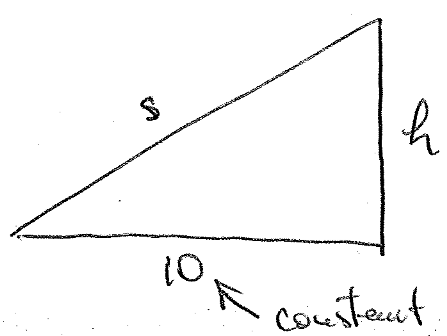
(c) If a function $f(x)$ is differentiable at $x = 2$ then it is continuous at $x = 2$. **True** **False**

Justification: Differentiability implies continuity

(d) If $p(x)$ is a polynomial of degree 10, then its eleventh derivative, $p^{(11)}(x) = 0$. True False

Justification: After every derivative, the degree drops by one, so the 10th derivative will be a constant. The 11th derivative must be identically 0.

3. (10 pts) A rocket, rising vertically, is tracked by a radar station that is on the ground 10 miles from the launchpad. How fast is the rocket rising when it is 8 miles high and its distance from the radar station is increasing at a rate of 1 mile/second ?



s, h are varying with time (t).

At the moment when $h = 8$ and $\frac{ds}{dt} = 1$, find $\frac{dh}{dt} = ?$

$s^2 = 10^2 + h^2$ (Pythagora)

Differentiate both sides w.r.t. time.

$$(s^2)' = (10^2)' + (h^2)'$$

$$2s\frac{ds}{dt} = 2h\frac{dh}{dt} \quad \text{so} \quad \frac{dh}{dt} = \frac{s}{h}\frac{ds}{dt}$$

when $h = 8$, $s^2 = 10^2 + 8^2 \Rightarrow s = \sqrt{164}$

so $\frac{dh}{dt} = \frac{\sqrt{164}}{8}\ \frac{mi}{s} \approx 1.6\ \frac{mi}{s}$ (as $\sqrt{169} = 13$, I approximated $\sqrt{164} \approx 12.8$ so $\frac{12.8}{8} = 1.6$)

4. (10 pts) Show that $y = x\cos(2x)$ is a solution for the differential equation $y'' + 4y = -4\sin(2x)$.

$$y' = (x\cos(2x))' = 1\cdot\cos(2x) + x(-\sin(2x))\cdot 2$$

$$y' = \cos(2x) - 2x\sin(2x)$$

$$y'' = -\sin(2x)\cdot 2 - (2\sin(2x) + 2x\cos(2x)\cdot 2)$$

$$y'' = -2\sin(2x) - 2\sin(2x) - 4x\cos(2x)$$

So $y'' = -4\sin(2x) - 4y$

thus $y'' + 4y = -4\sin(2x)$

so $y = x\cos(2x)$ is a solution of the given differential equation

5. (12 pts) A manufacturer of athletic footwear finds that the sales of their Xtride brand of running shoes is a function $f(p)$ of the selling price p (in dollars) for a pair of shoes. Suppose that $f(120) = 5000$ pairs of shoes and $f'(120) = -50$ pairs of shoes per dollar. The revenue that the manufacturer will receive for selling $f(p)$ pairs of shoes at p dollars per pair is $R(p) = p \cdot f(p)$.

(a) (8 pts) Find $R'(120)$.

product rule

$$R'(p) = (p\cdot f(p))' = 1\cdot f(p) + p f'(p)$$

$$R'(120) = f(120) + 120\cdot f'(120) = 5000 + 120\cdot(-50)$$

$$R'(120) = 5000 - 6000 = -1000$$

(b) (4 pts) In one sentence, explain the practical meaning of the result in part (a).

$R'(p) = \frac{dR}{dp}$ is the rate of change of revenue with respect to price

so $R'(120) = -1000$ means

If the manufacturer increases the price per pair of shoes from 120 to 121 then the revenue will decrease by approximately 1000\$.

No contradiction above because an increase in price leads to a decrease in the # of shoes sold.

6. (14 pts) (a) (8 pts) Use implicit differentiation to find dy/dx for the (rotated) ellipse $x^2 - xy + 3y^2 = 11$.

$$(x^2 - xy + 3y^2)' = (11)'$$

$$2x - 1 \cdot y - x \cdot y' + 6y \cdot y' = 0$$

$$2x - y = xy' - 6y\,y'$$

$$2x - y = (x - 6y)\,y'$$

so $\boxed{\dfrac{dy}{dx} = \dfrac{2x - y}{x - 6y}}$

(b) (6 pts) Find all points on the ellipse $x^2 - xy + 3y^2 = 11$ where the tangent line is horizontal.

Tangent line is horizontal when $\dfrac{dy}{dx} = 0$

This means ~~that~~ $2x - y = 0$ so $y = 2x$

But the point must be on the curve $x^2 - xy + 3y^2 = 11$

So we have the system $\begin{cases} y = 2x \\ x^2 - xy + 3y^2 = 11 \end{cases}$

Substituting, we get

$x^2 - x \cdot (2x) + 3(2x)^2 = 11$ so $11x^2 = 11 \Rightarrow x^2 = 1$

so $x = \pm 1$ hence $y = 2x = \pm 2$

The two points where tangent line is horizontal are $\boxed{(1, 2) \text{ and } (-1, -2)}$

7. (10 pts) Choose ONE:

(a) State and prove the formula for the derivative of a product of two functions.

(b) Find, with proof, the formula for $(\cos x)'$.

See class notes or textbook

For (a), either the proof with the limit definition of derivative or the proof with logarithmic differentiation are acceptable.