Name: Solution Key

Panther ID: ____________

Exam 3 Calculus I Fall 2016

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work may receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) These are True or False questions. In each case, circle your answer. No justification is necessary for this problem.

(a) (2 pts) If $f'(5) = 0$ and $f''(5) < 0$ then $f(x)$ has a relative maximum at $x = 5$ **True** **False**

(b) (2 pts) $0^{+\infty}$ is an indeterminate form for limits. **True** **False**

(c) (2 pts) $\int [f(x)-3g(x)]\,dx = \int f(x)\,dx - 3\int g(x)\,dx$ **True** **False**

(d) (2 pts) If $f(x)$ is continuous on $(0, 4)$, then f has an absolute minimum on $(0, 4)$. **True** **False**

(e) (2 pts) If $f'(x) < 0$ for all $x \in [0, 4]$, then, on the interval $[0, 4]$, $f(x)$ has an absolute minimum at $x = 4$. **True** **False**

2. (16 pts) Find the following limits:

(a) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ $= \lim_{x\to 0} \frac{e^x - 1 - x}{x\cdot(e^x-1)} \overset{l'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{e^x - 1}{1\cdot(e^x-1)+x\cdot e^x}$

$\infty - \infty$

$= \lim_{x\to 0} \frac{e^x-1}{xe^x+e^x-1} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \frac{e^x}{1\cdot e^x + xe^x + e^x} = \boxed{\frac{1}{2}}$

(b) $\lim_{x\to+\infty} x^{1/x}$ $= \lim_{x\to+\infty} e^{\ln(x^{\frac{1}{x}})} = \lim_{x\to+\infty} e^{\frac{\ln x}{x}}$

∞^0

$= e^{\lim_{x\to+\infty} \frac{\ln x}{x}} = e^0 = \boxed{1}$

$\lim_{x\to+\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}} \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$

3. (10 pts) (a) (6 points) Find the local linear approximation of the function $f(x) = \sqrt[3]{x}$ at $x_0 = 8$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}}, \text{ so } f'(8) = \frac{1}{3}\cdot 8^{-\frac{2}{3}} = \frac{1}{3\cdot(\sqrt[3]{8})^2} = \frac{1}{12}$$

Thus, $\boxed{\sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8)}$

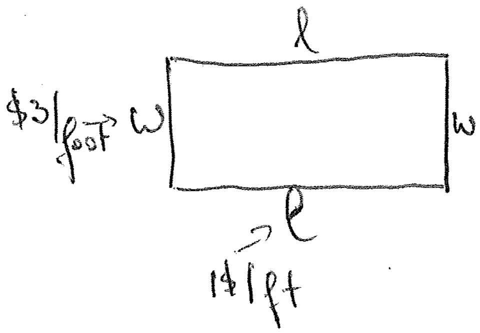
(b) (4 pts) Use the result of part (a) to approximate $\sqrt[3]{7.94}$.

OK like this

$$\sqrt[3]{7.94} \approx 2 + \frac{1}{12}(7.94-8) = 2 - \frac{0.06}{12} = 2 - 0.005 = 1.995$$

so $\sqrt[3]{7.94} \approx 1.995$

4. (14 pts) A rectangular area of 1200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$3 per foot. Find the dimensions of the rectangle of least cost.



$A = l\cdot w = 1200 \Longrightarrow l = \frac{1200}{w}$

C = total cost

$C = (2l)\cdot 1 + (2w)\cdot 3$

So $C(w) = 2\cdot\frac{1200}{w} + 6w = \frac{2400}{w} + 6w$

domain of $C(w)$: $w \in (0, +\infty)$

~~We need to minimize~~ $\cancel{C(w) = \frac{2400}{w} +}$

$$C'(w) = -\frac{2400}{w^2} + 6$$

$C'(w) = 0 \iff 6 = \frac{2400}{w^2} \iff w^2 = 400$ so $w = \sqrt{400} = 20$

$w = 20$ is an absolute minimum for the cost

since $\lim_{w\to 0} C(w) = \lim_{w\to +\infty} C(w) = +\infty$

can be seen also by computing second derivative and observing that $C''(w) > 0$.

The dimensions of the rectangle of least cost are $\boxed{w = 20,\ l = \frac{1200}{20} = 60}$

5. (12 pts) Consider $f(x) = x + \frac{1}{x}$.

(a) Find the absolute maximum and minimum values of f, if any, on $[\frac{1}{2}, 3]$.

(b) Find the absolute maximum and minimum values of f, if any, on $(0, +\infty)$.

$$f'(x) = 1 - \frac{1}{x^2}$$

so critical points are $x = \pm 1$

Only $x=1$ is in the intervals considered.

For part (a) we should compute

$$f(\tfrac{1}{2}) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{1}{2} + 2 = 2.5$$

$$f(1) = 1 + \frac{1}{1} = 2 \leftarrow \text{Absolute minimum (on } [\tfrac{1}{2}, 3])$$

$$f(3) = 3 + \frac{1}{3} = \frac{10}{3} \leftarrow \text{Absolute maximum (on } [\tfrac{1}{2}, 3])$$

For part (b),

$$f(1) = 2$$

and we compute $\lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} \left(x + \frac{1}{x}\right) = +\infty$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \left(x + \frac{1}{x}\right) = +\infty$$

Thus $f(1) = 2$ is the absolute minimum value of f on $(0, +\infty)$. There is no absolute maximum for the function on this interval.

6. (24 pts) Compute the following integrals (6 pts each):

(a) $\int \left(3^x + \frac{1}{1+x^2} - 2\right)\, dx =$

$= \frac{3^x}{\ln 3} + \arctan x - 2x + c$

(b) $\int \frac{2x+\sqrt{x}}{x^2}\, dx = \int \left(\frac{2x}{x^2} + \frac{x^{\frac{1}{2}}}{x^2}\right) dx$

$= \int \left(\frac{2}{x} + \frac{1}{x^{\frac{3}{2}}}\right) dx$

$= 2\int \frac{1}{x} dx + \int x^{-\frac{3}{2}} dx$

$= 2 \ln|x| - 2 \cdot x^{-\frac{1}{2}} + c$

$= 2 \ln|x| - \frac{2}{\sqrt{x}} + c$

(c) $\int \sin(3x) e^{\cos 3x}\, dx =$

sub $w = \cos(3x)$

$dw = -\sin(3x) \cdot 3 dx$

$-\frac{1}{3} dw = \sin(3x) dx$

$= \int e^w \cdot \left(-\frac{1}{3} dw\right) =$

$= -\frac{1}{3} \int e^w dw = -\frac{1}{3} e^w + c$

$\boxed{= -\frac{1}{3} e^{\cos(3x)} + c}$

(d) $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\, dx =$

sub.

$u = \ln x$

$du = \frac{1}{x} dx$

$= \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c$

$\boxed{= \arcsin(\ln x) + c}$

7. (24 pts) Consider the function $f(x) = \frac{x^2}{x^2-1} = \frac{x^2}{(x-1)(x+1)}$

Note is an even function, so the graph is symmetric w.r.t. y-axis.

(a) Find vertical and horizontal asymptotes of f, if any;

(b) Use a sign chart to find the intervals on which f is increasing; on which f is decreasing. Find critical points, if any, and determine the type of critical points (relative minimum, relative maximum or neither).

(c) Compute f'' and find the intervals on which f is concave up; on which f is concave down. Find the coordinates of all inflection points (if any).

(d) Sketch the graph of the function, showing all the information you have gathered in (a)-(c).

(a) Domain for $f(x)$: all reals, except $x = \pm 1$

$x=1$, $x=-1$ are vertical asymptotes (e.g. $\lim_{x \to 1^+} \frac{x^2}{(x-1)(x+1)} = \frac{1}{0^+} = +\infty$)

As $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1 \Rightarrow$ $y=1$ is a horiz. asymptote both when $x \to +\infty$ and $x \to -\infty$.

(b) $f'(x) = \left(\frac{x^2}{x^2-1}\right)' = \frac{2x(x^2-1) - x^2 \cdot 2x}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2} = \boxed{-\frac{2x}{(x-1)^2(x+1)^2}}$

Thus, $f'(x) = 0 \Leftrightarrow x = 0 \leftarrow$ the only critical point

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		$+ + + + + +$	‖	$+ + +$	0	$- -$	‖	$- - - - - -$	
$f(x)$	1	$\nearrow$	$+\infty$ ‖ $-\infty$	$\nearrow$	0	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	1
$f''(x)$		$+ + + + +$	‖	$- - - -$			‖	$+ + + +$	

The intervals over which f is increasing are $(-\infty,-1) \cup (-1,0)$

f is decreasing on $(0,1) \cup (1,+\infty)$

$(0,0)$ is a relative maximum

(c) $f''(x) = \left(-\frac{2x}{(x^2-1)^2}\right)' = -\frac{2(x^2-1)^2 - 2x \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4}$

$f''(x) = -\frac{2(x^2-1)[x^2-1-4x^2]}{(x^2-1)^4} = -\frac{2(-3x^2-1)}{(x^2-1)^3}$

so $f''(x) = \frac{2(3x^2+1)}{(x-1)^3(x+1)^3}$

f is concave up on $(-\infty,-1) \cup (1,+\infty)$

f is concave down on $(-1,1)$

There are no inflection points, because changes in concavity occur only at $x=+1$, $x=-1$ which are not in the domain of f. (see sign chart above for f''.)

(d)