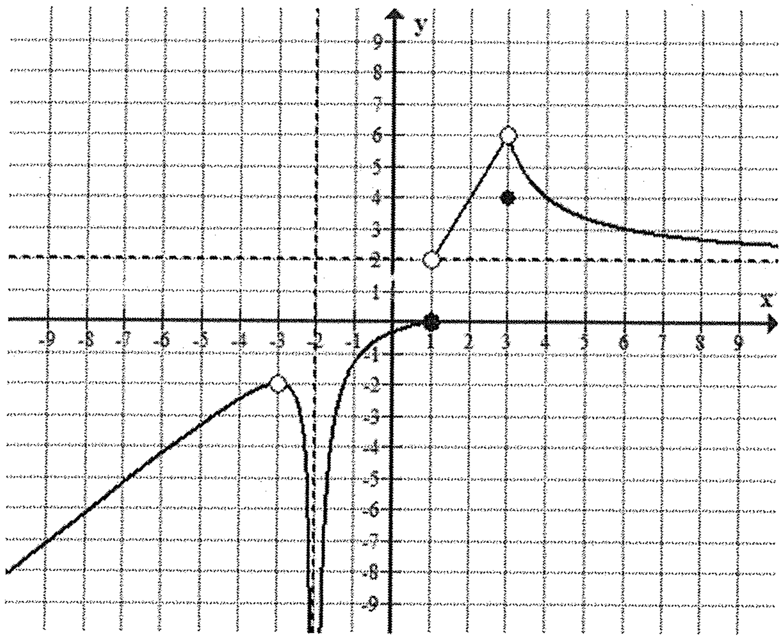
Name: Solution Key PID: ____________

Fall 2016 -- MAC 2311- Exam 1

There are 7 problems for a total of 110 points. **Show your work**; an answer alone, even correct, will get no credit. An illegible solution will not be graded. Organize your work so it is clear what you do and why. **Calculators are not allowed.**

Problem 1.(15 pts) Use the graph of the function f given below to answer the questions that follow.



(i) (9 pts) Find the following limits (you don't have to show any work here)

a) $\lim_{x\to-\infty} f(x) = -\infty$ b) $\lim_{x\to+\infty} f(x) = 2$ c) $\lim_{x\to-2} f(x) = -\infty$

d) $\lim_{x\to1^-} f(x) = 0^-$ e) $\lim_{x\to1^+} f(x) = 2$ f) $\lim_{x\to1} f(x) =$ D.N.E

g) $\lim_{x\to4} f(x) = 4$ h) $\lim_{x\to3} f(x) = 6$ i) $\lim_{x\to-3} f(x) = -2$

(ii) (4 pts) Is f continuous everywhere? If not, give x value(s) at which f has a discontinuity. Specify if any of the discontinuities is removable.

- f is not continuous everywhere.
- There is a removable discontinuity at $x = -3$ and 3
- Other discontinuities at $x = -2, 1$

(iii) (2 pts) Is f continuous on the interval [2,6]? Answer and briefly justify.

- f is not continuous on the interval [2,6] because there is a discontinuity at $x = 3$. It does not satisfy the definition of discontinuity. $\left(\lim_{x\to3} f(x) \neq f(3)\right)$ So its not a function
- It's a removable discontinuity

Problem 2. (36 pts) Find the following limits. Show all work and explain clearly (6 pts each).

a) $\lim\limits_{x\to 3^+} \dfrac{x}{3-x} =$ $\left(\dfrac{3}{3-3} = \dfrac{3}{0} = \infty \right)$

- $\text{limit}_{x\to 3^+} \dfrac{x}{3-x} = -\infty$
- Because $3-x$ will be a small negative number since x is approaching 3 from the right. $(3-3.0001 = -0.0001)$

b) $\lim\limits_{x\to -1} \dfrac{3x+3}{x^2+4x+3} =$ $\dfrac{3(-1)+3}{(-1)^2+4(-1)+3} = \dfrac{0}{0}$ indeterminate form

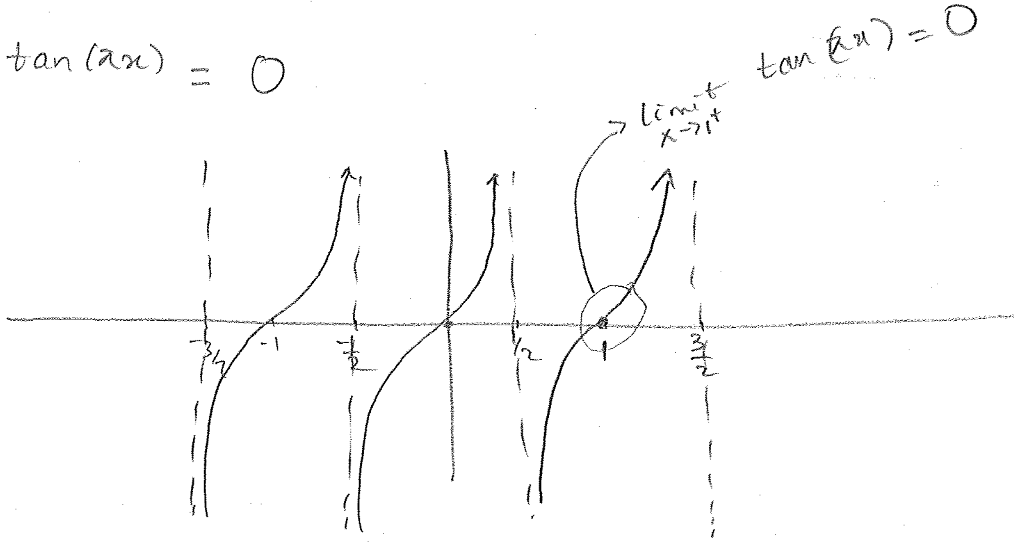
- $\text{limit}_{x\to -1} \dfrac{3\cancel{(x+1)}}{\cancel{(x+1)}(x+3)} = \text{limit}_{x\to -1} \dfrac{3}{x+3}$
- $\Rightarrow \boxed{\dfrac{3}{2}}$

c) $\lim\limits_{x\to 2} \dfrac{2x-4}{\sqrt{x+7}-3} =$ $\dfrac{2(2)-4}{\sqrt{2+7}-3} = \dfrac{0}{0}$ indeterminate form.

- $\text{Limit}_{x\to 2} \dfrac{2x-4}{\sqrt{x+7}-3} \cdot \dfrac{(\sqrt{x+7}+3)}{(\sqrt{x+7}+3)} = \text{limit}_{x\to 2} \dfrac{(2x-4)(\sqrt{x+7}+3)}{x+7-9}$
- $\text{limit}_{x\to 2} \dfrac{2\cancel{(x-2)}(\sqrt{x+7}+3)}{\cancel{(x-2)}} = \text{limit}_{x\to 2}\, 2(\sqrt{x+7}+3)$
- $\boxed{12}$

d) $\lim_{x\to 1^+} \tan(\pi x) =$

• $\lim_{x\to 1^+} \tan(\pi x) = 0$



e) $\lim_{x\to 0} \dfrac{\tan^2(4x)}{x\sin(5x)} =$

• $\lim_{x\to 0} \dfrac{\tan(4x) \cdot \tan(4x) \cdot \frac{(4x)^2}{(4x)^2}}{x \cdot \sin(5x) \cdot \frac{5x}{5x}}$

• $\lim_{x\to 0} \dfrac{\cancel{\frac{\tan(4x)}{(4x)}}^{1} \cdot \cancel{\frac{\tan(4x)}{(4x)}}^{1} \cdot 16\cancel{x^2}}{5\cancel{x^2} \; \cancel{\frac{\sin(5x)}{5x}}^{1}} = \lim_{x\to 0} \boxed{\dfrac{16}{5}}$

• $\boxed{\dfrac{16}{5}}$

f) $\lim_{t\to -\infty} \dfrac{\sqrt{1+2t^2}}{2+3t} =$ $\dfrac{\sqrt{1+2(-\infty)^2}}{2+3(-\infty)} = \dfrac{\infty}{\infty}$ indeteminate form

• $\lim_{t\to -\infty} \dfrac{\sqrt{1+2t^2} \cdot \frac{1}{|t|}}{2+3t \cdot \frac{1}{|t|}} = \lim_{t\to -\infty} \dfrac{\sqrt{1+2t^2} \cdot \frac{1}{\sqrt{t^2}}}{\frac{2}{|t|} + \frac{3t}{|t|}} = \lim_{t\to -\infty} \dfrac{\sqrt{\frac{1}{t^2} + \frac{2t^2}{t^2}}}{\frac{2}{-t} + \frac{3\cancel{t}}{-\cancel{t}}}$

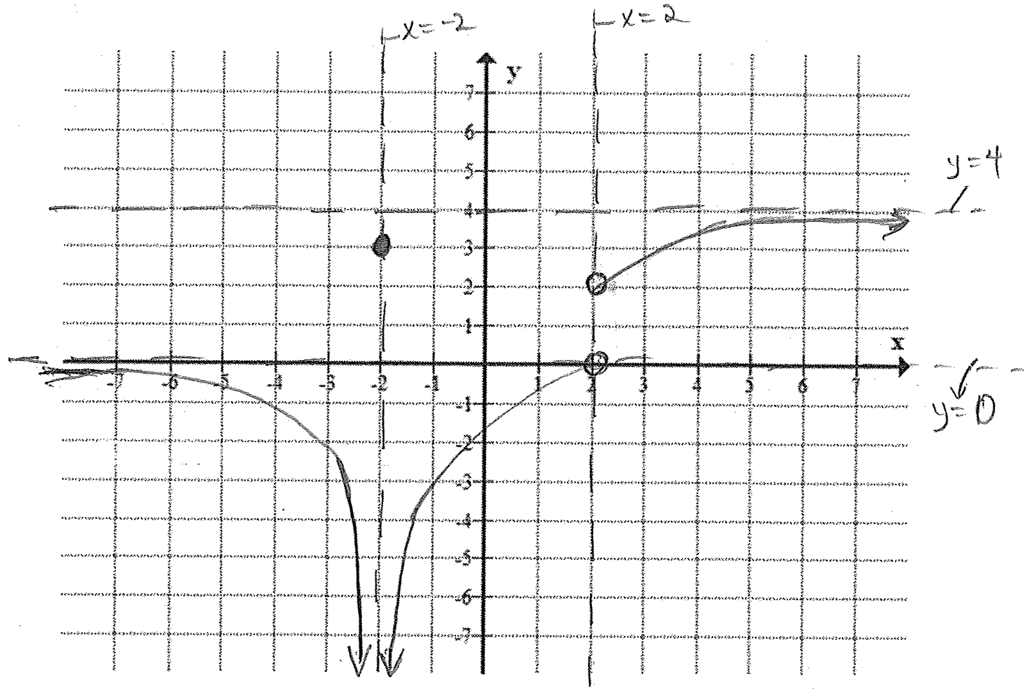
• $\Rightarrow \lim_{t\to -\infty} \dfrac{\sqrt{\cancel{\frac{1}{t^2}}^{0} + 2}}{\cancel{\frac{-2}{t}}^{0} - 3} = \dfrac{\sqrt{2}}{-3} = \boxed{\dfrac{-\sqrt{2}}{3}}$ OR

• Junk rule $\lim_{t\to -\infty} \dfrac{\sqrt{2}\,|t|}{3t} = \lim_{t\to -\infty} -\dfrac{\sqrt{2}\,\cancel{t}}{3\cancel{t}} = \boxed{\dfrac{-\sqrt{2}}{3}}$

NOTE: $|t|$ as $t \to -\infty = -t$

Problem 3. (12 pts) Sketch the graph of a function f satisfying ALL of the following properties:

a) The domain of f is the set of all real numbers except x = 2;
b) The function is continuous at all points except x = -2 and x = 2;
c) $\lim_{x\to -2} f(x) = -\infty$, f(-2) = 3;
d) $\lim_{x\to 2^-} f(x) = 0$, $\lim_{x\to 2^+} f(x) = 2$;
e) $\lim_{x\to -\infty} f(x) = 0$;
f) $\lim_{x\to +\infty} f(x) = 4$.



Problem 4. (10 pts) (a) (3 pts) Write the general (ε,δ) definition for $\lim_{x\to a} f(x) = L$.

Given any $\varepsilon > 0$ we can find $\delta > 0$ so that if $|x-a| < \delta$ and $x \neq a$ then $|f(x) - L| < \varepsilon$.

(b) (7 pts) Use the epsilon-delta definition to prove $\lim_{x\to -3} 10x + 3 = -27$.

In this case, $|f(x) - L| = |10x + 3 - (-27)| = |10x + 30| = 10|x+3|$

So given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{10}$.

Then if $|x-(-3)| < \delta$ then $|f(x) - L| = 10|x+3| < 10\delta = \cancel{10} \cdot \frac{\varepsilon}{\cancel{10}} = \varepsilon$

Problem 5. (15 pts) A stone is dropped from the top of a building. Its position s(t) in feet above the ground after t seconds is given by $s(t) = 160 - 16t^2$.

(a) (3 pts) When does the stone reach the ground?

- The stone hits the ground when height $= 0$;

  $s(t) = 0$; $160 - 16t^2 = 0$

- $16t^2 - 160 = 0 \longrightarrow 16(t^2 - 10) = 0$
- $t^2 - 10 = 0 \rightarrow t^2 = 10$
- $t = \pm\sqrt{10}$; Stone hits the ground at $t = \sqrt{10}$ Secs

(b) (5 pts) Find the average velocity of the stone during the first two seconds of its fall.

- Average Velocity for first two seconds $= \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{s(2) - s(0)}{2}$

- $s(2) = 160 - 16(2)^2 = 160 - 16(4) = 160 - 64 = 96$ ft
- $s(0) = 160 - 16(0^2) = 160$ ft

$\therefore V_{avg} = \dfrac{96 - 160}{2} = \dfrac{-64 \text{ ft}}{2 \text{ s}}$

- $V_{avg} = -32$ ft/s

(b) (7 pts) Use limits to find the instantaneous velocity of the object at 2 seconds.

- Instantaneous velocity at 2 Seconds $\Rightarrow \lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$

- $s(2+h) = 160 - 16(2+h)^2 = 160 - 16[4 + 4h + h^2]$
- $s(2+h) = 160 - 64 - 64h - 16h^2 = (96 - 64h - 16h^2)$ ft
- $s(2) = 160 - 16(2)^2 = 160 - 64 = 96$ ft

- $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(t)}{h} = \lim\limits_{h \to 0} \dfrac{\cancel{96} - 64h - 16h^2 - (\cancel{96})}{h}$

- $\lim\limits_{h \to 0} \dfrac{-64h - 16h^2}{h} = \lim\limits_{h \to 0} \dfrac{\cancel{h}(-64 - \cancel{16h}^{0})}{\cancel{h}}$

Instantaneous Velocity $= -64$ ft/s

Problem 6. (12 pts) (a) (6 pts) Use the Intermediate Value Theorem to show that the equation $x^3 = 9x - 1$ has a solution in the interval [0,1]. Explain thoroughly.

$x^3 = 9x - 1 \iff x^3 - 9x + 1 = 0$

Let $f(x) = x^3 - 9x + 1$. It is a polynomial, so it is continuous everywhere. So I.V.T. applies to this function on any interval.

$f(0) = 1 > 0$
$f(1) = 1 - 9 + 1 = -9 < 0$ $\xRightarrow{\text{By I.V.T.}}$ there is a point $x_1 \in (0,1)$ so that $f(x_1) = 0$. This means that $x^3 - 9x + 1 = 0$ has a root $x_1 \in (0,1)$.

(b) (6 pts) Use the Intermediate Value Theorem to show that the equation $x^3 = 9x - 1$ has three distinct solutions. Explain thoroughly. We know there is a solution x_1 on the interval $(0,1)$

We test other values

$f(2) = 2^3 - 9 \cdot 2 + 1 < 0$
$f(3) = 3^3 - 9 \cdot 3 + 1 > 0$ $\xRightarrow{\text{By I.V.T.}}$ there is another solution $x_2 \in (2,3)$ (that is, $f(x_2) = 0$)

Also $f(-3) = (-3)^3 - 9 \cdot (-3) + 1 = 1 > 0$
but $f(-4) = (-4)^3 - 9 \cdot (-4) + 1 < 0$ $\Rightarrow$ there is a 3rd solution $x_3 \in (-4, -3)$

We proved that the given equation has three distinct real roots.

Problem 7. (10 pts) Choose ONE of the following:

(a) Assuming the inequality $\sin x \le x \le \tan x$ for any $x \in \left[0, \frac{\pi}{2}\right)$, prove that $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$.

(b) Find the limit $\lim_{x \to +\infty} \frac{\cos(3x)}{x}$ and justify your answer.

For (a) see notes or textbook

For (b), (also) apply the Squeeze Theorem.

$-1 \le \cos(3x) \le 1 \quad | \div x \quad x > 0$

$-\frac{1}{x} \le \frac{\cos(3x)}{x} \le \frac{1}{x}$

$x \to +\infty$: each part $\to 0$

Since $\lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to +\infty} \left(-\frac{1}{x}\right) = 0$,

it follows that

$$\boxed{\lim_{x \to +\infty} \frac{\cos(3x)}{x} = 0}$$