Name: Solution Key Panther ID: ____________

Exam 1 - MAC2311 - Spring 2016

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) These are True or False questions. Circle your answer (2 pts) and briefly justify (2 pts).

(a) If $\lim_{x\to 3} f(x) = 6$ and $\lim_{x\to 3} g(x) = -1$ then $\lim_{x\to 3}(f(x)+2g(x)) = 4$ **True** **False**

Justification: Arithmetic properties of limits

(b) The function $f(x) = \cot x$ is defined and is continuous for all real numbers x. **True** **False**

Justification: $\cot x = \frac{\cos x}{\sin x}$ is not defined at all points where $\sin x = 0$. That is, it's not defined at $x = k\cdot\pi$, with k integer.

(c) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ **True** **False**

Justification: $\frac{0}{0}$ is an indeterminate form which could lead to any result

(d) The equation $x^3-8x+1 = 0$ has a real solution in the interval [1,2]. **True** **False**

Justification: We try to apply I.V.T. to the function $f(x) = x^3-8x+1$ on the interval $[1,2]$. ($f(x)$ is continuous since it's a polynomial)
We compute
$f(1) = 1-8+1 = -6 < 0$ $\quad f(2) = 2^3-8\cdot 2+1 = -7 < 0$
Thus we cannot draw any conclusion from I.V.T. if the equation $x^3-8x+1=0$ has a solution or not in the interval $[1,2]$
We cannot, at this point, decide if the answer to this question is True or False. Later, we'll see that the answer is indeed False.

2. (40 pts) Find the following limits (5 pts each). If the limit is infinite or does not exist, specify so.

(a) $\lim_{x\to 2} \frac{3x-6}{x^2-5x+6} \overset{\frac{0}{0}}{=}$

$= \lim_{x\to 2} \frac{3\cancel{(x-2)}}{\cancel{(x-2)}(x-3)} = \boxed{-3}$

(b) $\lim_{x\to -3^-} \frac{1+2x}{x+3} = \frac{-5}{0^-} = \boxed{+\infty}$

(c) $\lim_{t\to 2} \frac{|t-2|}{t^2-4}$

$|t-2| = \begin{cases} t-2 & \text{if } t>2 \\ -(t-2) & \text{if } t<2 \end{cases}$

so $\lim_{t\to 2^+} \frac{|t-2|}{t^2-4} \underset{t>2}{=} \lim_{t\to 2^+} \frac{\cancel{t-2}\ 1}{\cancel{(t-2)}(t+2)} = \frac{1}{4}$

$\lim_{t\to 2^-} \frac{|t-2|}{t^2-4} \underset{t<2}{=} \lim_{t\to 2^-} \frac{-\cancel{(t-2)}}{\cancel{(t-2)}(t+2)} = -\frac{1}{4}$

Thus, $\lim_{t\to 2} \frac{|t-2|}{t^2-4}$ D.N.E.

(d) $\lim_{x\to -\infty} \frac{x-4x^5}{1+2x^2+3x^4}$

Jumbo Rule

$= \lim_{x\to -\infty} \frac{-4x^{\cancel{5}}}{3x^{\cancel{4}}} = \lim_{x\to -\infty} \frac{-4x}{3} = \boxed{+\infty}$

(e) $\lim_{x\to 0} \frac{\tan(5x)}{x+\sin(2x)} \overset{\frac{0}{0}}{=}$

$= \lim_{x\to 0} \frac{5x \cdot \frac{\tan(5x)}{5x}}{x + 2x \cdot \frac{\sin(2x)}{2x}} =$

$= \lim_{x\to 0} \frac{5\cancel{x} \cdot \frac{\tan(5x)}{5x}}{\cancel{x}\left[1 + 2\cdot \frac{\sin(2x)}{2x}\right]}$

$= \frac{5\cdot 1}{1+2} = \boxed{\frac{5}{3}}$

(f) $\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{x+2} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to -\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(1+\frac{2}{x}\right)}$

$= \lim_{x\to -\infty} \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(1+\frac{2}{x}\right)} =$

$\underset{x<0}{=} \lim_{x\to -\infty} \frac{-\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(1+\frac{2}{x}\right)} = \boxed{-\sqrt{2}}$

(g) $\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \frac{0}{0}$

$$= \lim_{x\to 0} \frac{(1-\cos(3x))(1+\cos(3x))}{x^2(1+\cos(3x))}$$

$$= \lim_{x\to 0} \frac{\sin^2(3x)}{x^2(1+\cos(3x))} =$$

$$= \lim_{x\to 0} \frac{9\cancel{x^2}\cdot\frac{\sin^2(3x)}{(3x)^2}}{\cancel{x^2}(1+\cos(3x))} = \boxed{\frac{9}{2}}$$

(h) $\lim_{x\to +\infty} \frac{1-\cos(3x)}{x^2} = 0$

Squeeze Theorem:

$-1 \leq \cos(3x) \leq 1$ so

$1 \geq -\cos(3x) \geq -1$ | Add 1

$2 \geq 1-\cos(3x) \geq 0$

So

$$\frac{2}{x^2} \geq \frac{1-\cos(3x)}{x^2} \geq 0$$

$x \to +\infty$ $\Rightarrow$ 0

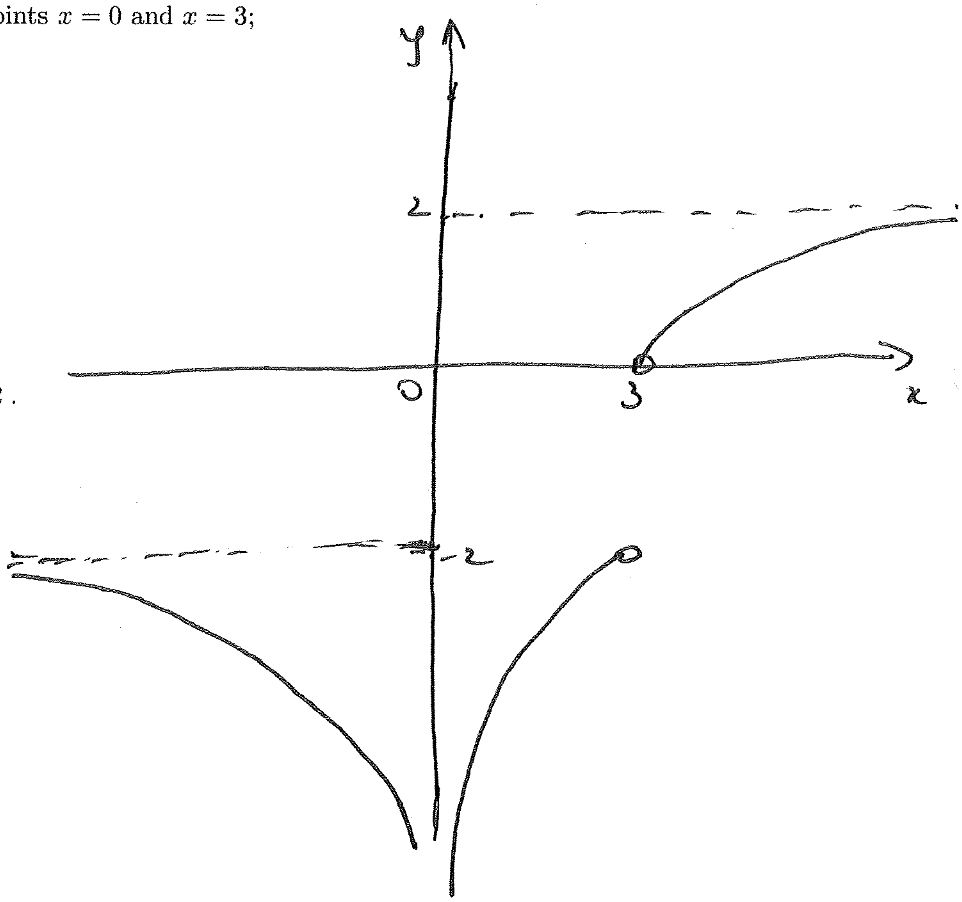
3. (12 pts) Sketch the graph of ONE function $f(x)$ satisfying ALL of the following conditions.

(i) The function is defined and continuous everywhere except $x=0$ and $x=3$;
The function is not defined at the points $x=0$ and $x=3$;

(ii) $\lim_{x\to 0} f(x) = -\infty$;

(iii) $\lim_{x\to 3^-} f(x) = -2$, $\lim_{x\to 3^+} f(x) = 0$;

(iv) $\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to +\infty} f(x) = 2$.



4. (10 pts) Use limits to find the slope of the tangent line to the graph of $f(x) = x^2 - 3x$ at $x = 3$.

$$m_{\substack{tan \\ \text{at } x_0=3}} = f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} =$$

$$= \lim_{h\to 0} \frac{[(3+h)^2 - 3\cdot(3+h)] - [\overset{0}{3^2 - 3\cdot 3}]}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9} - 3h}{h} = \lim_{h\to 0} \frac{3h + h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(3+h)}{\cancel{h}\,1}$$

$$= \lim_{h\to 0} \frac{3+h}{1} = \boxed{3} \quad \text{so} \quad \boxed{m_{\substack{tangent \\ \text{at } x=3}} = 3}$$

5. (10 pts) Given the function below

$$g(x) = \begin{cases} kx^2 - 1 & \text{if } x \le 1 \\ 2x + k & \text{if } x > 1 \end{cases}$$

(a) (5 pts) Is there a value of the constant k which will make the function continuous? Justify your answer.

The question is about continuity at $x = 1$.

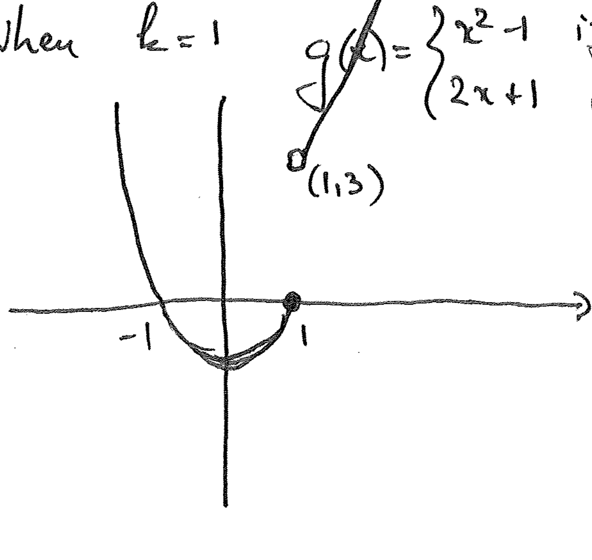
We want $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = g(1)$

But $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} (kx^2 - 1) = \underset{g(1)}{\underset{\shortparallel}{k-1}}$ $\quad \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (2x+k) = k+2$

We want $k - 1 = k + 2$ but this has no solution. Thus, there is no value of k which will make the function continuous at $x = 1$.

(b) (5 pts) Sketch the graph of the function $g(x)$ when $k = 1$.
Label carefully the coordinates of important points.

When $k = 1$ $\quad g(x) = \begin{cases} x^2 - 1 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1. \end{cases}$



6. (a) (3 pts) Write the general (ϵ, δ) definition for $\lim_{x \to a} f(x) = L$.

For every $\varepsilon > 0$, we can find $\delta > 0$ so that
if $|x-a| < \delta$ and $x \neq a$ then $|f(x) - L| < \varepsilon$

Choose ONE of the parts (b) and (c). Only ONE will receive credit. Note the different point values.

(b) (7 pts) Use the (ϵ, δ) definition to prove $\lim_{x \to -2} (5x+4) = -6$.

(c) (12 pts) Use the (ϵ, δ) definition to prove $\lim_{x \to 2} (2x^2+3) = 11$.

(b) In this case $|f(x) - L| = |(5x+4) - (-6)| = |5x+10| = 5 \cdot |x+2|$

So, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{5}$.

Then if $|x-(-2)| = |x+2| < \delta$ then $|f(x) - L| = 5 \cdot |x+2| < 5 \cdot \delta = \cancel{5} \cdot \frac{\varepsilon}{\cancel{5}}$

so we proved that $\lim_{x \to -2} (5x+4) = -6$

(c) We have $|f(x) - L| = |2x^2+3-11| = |2x^2-8| = 2 \cdot |x-2| \cdot |x+2|$

Preliminary choice: Let $0 < \delta \leq 1$.

Then if $|x-2| < \delta \leq 1 \Rightarrow |x-2| < 1 \Rightarrow -1 < x-2 < 1 \xRightarrow{\text{Add } 4} 3 < x+2 < 5$

Thus, we know that if $|x-2| < \delta \leq 1$ then $|x+2| < 5$. $(*)$

So, if $|x-2| < \delta \leq 1$ then $|f(x) - L| = 2 \cdot |x-2| \cdot |x+2| < 2 \cdot \delta \cdot 5 = 10\delta$

$|x-2| < \delta$ $|x+2| < 5$

Final choice for δ:

Given $\varepsilon > 0$, choose $\delta = \min\left(\frac{\varepsilon}{10}, 1\right)$. This choice satisfies $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{10}$

So if $|x-2| < \delta$ then

$|f(x) - L| = 2 \cdot |x-2| \cdot |x+2| < 2 \cdot \delta \cdot 5 = 10\delta \leq \cancel{10} \cdot \frac{\varepsilon}{\cancel{10}} = \varepsilon$

so we proved $\lim_{x \to 2} (2x^2+3) = 11$

7. (10 pts) Choose ONE of the following:

(a) State and prove the quadratic formula.

See notes or algebra textbook

(b) Using the inequality $\sin x \le x \le \tan x$ for any $x \in (0, \pi/2)$, show that $\lim_{x\to 0+} \frac{\sin x}{x} = 1$.

see notes or textbook