Name: Solution Key Panther ID: ____________

Worksheet week 4 Calculus I Spring 2016

1. Find, if possible, a value for the constant k which will make the function $g(x)$ continuous everywhere.

$$g(x) = \begin{cases} \frac{1-\cos(kx)}{x^2} & \text{if } x < 0 \\ 1+\sin(3x) & \text{if } x \geq 0 \end{cases}.$$

The only problem point might be $x=0$.

We want $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0)$

But $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{1-\cos(kx)}{x^2} = \lim_{x\to 0^-} \frac{(1-\cos(kx))(1+\cos(kx))}{x^2(1+\cos(kx))} =$

$= \lim_{x\to 0^-} \frac{\sin^2(kx)}{x^2(1+\cos(kx))} = \lim_{x\to 0} \frac{k^2\sin^2(kx)}{(kx)^2(1+\cos(kx))} = \frac{k^2}{2}$

Also, $g(0) = \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (1+\sin(3x)) = 1+\sin(0) = 1$

Thus, we want $\frac{k^2}{2} = 1$, so $k^2 = 2$, so $\boxed{k = \pm\sqrt{2}}$

2. (a) Use IVT to show that the equation $x^3 = 3x - 1$ has a solution in the interval $[0, 1]$.

(b) Approximate the solution in part (a) with an accuracy of 0.25; that is find an interval of length 1/4 which contains the solution.

(c) Use again IVT to show that the equation $x^3 = 3x - 1$ has three real solutions and find intervals of length 1 containing each solution.

(a) $x^3 = 3x-1$ is equivalent to solving $x^3-3x+1=0$.

Let $f(x) = x^3-3x+1$. This is continuous everywhere since it's a polynomial.

Since $f(0) = 1 > 0$ and $f(1) = -1 < 0$, by I.V.T., there is a point $x_1^* \in [0,1]$ so that $f(x_1^*) = 0$. So the equation $x^3-3x+1=0$ has a solution in $[0,1]$.

(b) We compute $f(0) = 1 > 0$, $f(\frac{1}{4}) = (\frac{1}{4})^3 - \frac{3}{4} + 1 > 0$, $f(\frac{1}{2}) = (\frac{1}{2})^3 - \frac{3}{2} + 1 < 0$

so again, by I.V.T., there is a solution in the interval $[\frac{1}{4}, \frac{1}{2}]$.

This is likely (but not ~~surely~~ at this point) the solution from part (a)

(c) We compute $f(0) = 1 > 0$, $f(1) = -1 < 0$, $f(2) = 3 > 0$,

$f(-1) = 3 > 0$, $f(-2) = -1 < 0$, so by I.V.T we conclude that the equation $x^3-3x+1=0$ has a real solution in each of the intervals $[0,1]$, $[1,2]$, $[-2,-1]$, thus we proved that the equation has three real solutions.

One can approximate each solution better & better with the technique from part (b).

3. Use the ε-δ definition of limit to prove that $\lim_{x\to 5}(2x+3) = 13$.

In general: $\lim_{x\to a} f(x) = L \iff$ for any $\varepsilon > 0$, we can find $\delta > 0$ so that if $\underset{x\neq a}{|x-a| < \delta}$ then $|f(x)-L| < \varepsilon$.

In our case, $|f(x)-L| = |2x+3-13| = |2x-10| = 2\cdot|x-5|$

Thus, given $\varepsilon > 0$,

$|f(x)-L| < \varepsilon \iff 2\cdot|x-5| < \varepsilon \iff |x-5| < \frac{\varepsilon}{2}$

So, choose $\delta = \frac{\varepsilon}{2}$.

Then, if $|x-5| < \delta \Rightarrow 2\cdot|x-5| < 2\delta = \cancel{2}\cdot\frac{\varepsilon}{\cancel{2}}$, so $|f(x)-13| < \varepsilon$

4. True or **False** questions. Answer and briefly justify your answer in each case.

(a) If $|f(x)+7| \le 3|x+2|$ for all real x, then $\lim_{x\to -2} f(x) = -7$.

True. With the definition above, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$.

Then, if $|x-(-2)| = |x+2| < \delta \Rightarrow |f(x)-(-7)| = |f(x)+7| \underset{\text{given}}{\le} 3|x+2| < 3\delta = \cancel{3}\frac{\varepsilon}{\cancel{3}}$

(b) If $f(x)$ is continuous at $x = 2$ and $f(2) = 5$, then for x sufficiently close to 2, $f(x) > 4.95$.

True: f continuous at $x=2$ implies $\lim_{x\to 2} f(x) = f(2) = 5$

Then for $\varepsilon = 0.05$ (take 0.05, which is $5-4.95$),

there is a $\delta > 0$ so that if

$|x-2| < \delta$ then $|f(x)-5| < 0.05$

But $|f(x)-5| < 0.05$ means $-0.05 < f(x)-5 < 0.05$, or

$4.95 < f(x) < 5.05$

Thus if x is in the interval $(2-\delta, 2+\delta)$, we get that $f(x) > 4.95$, so the statement is true