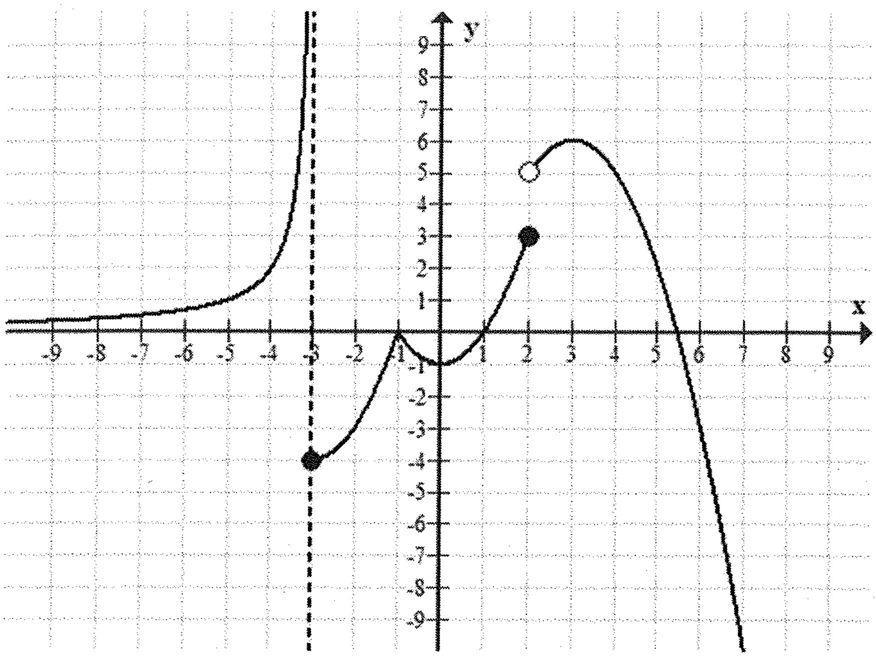
Name: Solution Key (from Olivera) PID: ____________

Spring 2017 -- MAC 2311- Exam 1 – Version A

There are 8 problems for a total of 110 points. **Show your work**; an answer alone, even correct, may get no credit. An illegible solution will not be graded. **Calculators are not allowed.**

Problem 1. (12pts) The graph of a function f is given below. Use the graph to answer the questions that follow.



(i) (7 pts) Find the following limits. You don't have to show any work for these, but specify if a limit is infinite or does not exist.

$\lim_{x\to -3^-} f(x) = +\infty$ $\lim_{x\to -3^+} f(x) = -4$ $\lim_{x\to -3} f(x) =$ DNE

$\lim_{x\to 2} f(x) =$ DNE (two sided limits are not the same) $\lim_{x\to 0} f(x) = -1$

$\lim_{x\to -\infty} f(x) = 0$ $\lim_{x\to +\infty} f(x) = -\infty$

(ii) (3 pts) Is f continuous everywhere? If not, give x value(s) at which f has a discontinuity. Specify if any of the discontinuities is removable.

f is not continuous. Two discontinuities at $x=-3$ and $x=2$, both not removable

(iii) (2 pts) Identify any point(s) x, where the function is continuous, but it is not differentiable. Specify if there is no such point x.

at $x=-1$ the function is continuous but not differentiable because it has cusp (i.e. the two sided limit will have ~~opposite~~ different sign)

Problem 2. (30 pts) Find the following limits. Specify if a limit is infinite or does not exist. Show all work and explain clearly (5 pts each).

a) $\lim_{x \to -4^-} \frac{x}{x+4} = \frac{-4}{0^-} = \boxed{+\infty}$

b) $\lim_{x \to 3} \frac{2x^2 - 18}{x^2 - x - 6} =$ $\frac{0}{0}$ form $\lim_{x \to 3} \frac{2(x^2-9)}{(x-3)(x+2)} = \lim_{x \to 3} \frac{2\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+2)} = \frac{2(3+3)}{5} = \boxed{\frac{12}{5}}$

c) $\lim_{x \to -3} \frac{x+3}{\sqrt{7+x} - 2} =$ $\frac{0}{0}$ form $\lim_{x \to -3} \frac{x+3}{\sqrt{7+x} - 2} \cdot \frac{\sqrt{7+x}+2}{\sqrt{7+x}+2} = \lim_{x \to -3} \frac{(x+3)(\sqrt{7+x}+2)}{7+x-4} =$

$\lim_{x \to -3} \frac{\cancel{(x+3)}(\sqrt{7+x}+2)}{\cancel{(x+3)}} = \sqrt{7-3} + 2 = \boxed{4}$

d) $\lim_{x\to+\infty} \sin\left(\frac{2\pi x}{4x+1}\right) = \cancel{\lim_{x\to+\infty}}\ \sin\Big(\underbrace{\lim_{x\to+\infty} \frac{2\pi x}{4x+1}}_{\text{garbage rule}}\Big) = \sin\left(\lim_{x\to+\infty} \frac{2\pi \cancel{x}}{\cancel{x}(4+\frac{1}{x})}\right) =$

$\frac{1}{x} \to 0$

$= \sin\left(\frac{\pi}{2}\right) = \boxed{1}$

e) $\lim_{x\to 0} \frac{x\tan(3x)}{\sin^2(5x)} = \lim_{x\to 0} \frac{x}{\sin^2(5x)}\left(\frac{5^2x}{5^2x}\right) \cdot \frac{\tan(3x)}{3x} \cdot 3x =$

$= \lim_{x\to 0} \frac{(5x)^2}{\sin^2(5x)} \cdot \frac{\tan(3x)}{3x} \cdot \frac{3\cancel{x}}{5^2\cancel{x}} = \boxed{\frac{3}{25}}$

$\frac{(5x)^2}{\sin^2(5x)} \to 1 \qquad \frac{\tan(3x)}{3x} \to 1$

Special limits

f) $\lim_{t\to-\infty} \frac{\sqrt{2t^2-t+1}}{5t} = \lim_{t\to-\infty} \frac{\sqrt{t^2(2-\frac{1}{t}+\frac{1}{t^2})}}{5t} = \lim_{t\to-\infty} \frac{|t|\sqrt{2-\frac{1}{t}+\frac{1}{t^2}}}{5t} =$

$= \lim_{t\to-\infty} \frac{-\cancel{t}\sqrt{2-\cancel{\frac{1}{t}}^{\,0}+\cancel{\frac{1}{t^2}}^{\,0}}}{5\cancel{t}} = \boxed{\frac{-\sqrt{2}}{5}}$

Problem 3. (12 pts) These are true or false questions. Answer (1pt) and give brief justification (2pts). Graph can serve as a justification.

(a) A function can never cross its horizontal asymptote. **True** **False**

Justification: def. of horizontal asym. is how the function behaves when x goes to $\pm\infty$. But before x gets too big the function can cross that asymptote many times.

(b) If $f(x)$ is continuous everywhere then $|f(x)|$ must be continuous everywhere. **True** **False**

Justification: $f(x)$ continuous means $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x) = f(a)$ for all a in the domain of $f(x)$.

but $\lim_{x\to a^+} f(x) = \lim_{x\to a} |f(x)| = f(a) \Rightarrow |f(x)|$ is continuous as well

(c) If a function f is continuous at $x=0$, then f is differentiable at $x=0$. **True** **False**

Justification: counterexample $f(x) = |x|$

Not differentiable at $x=0$ because of the cusp.

(d) If a function satisfies $|f(x) - 5| \leq 7|x-3|$ for all real numbers x, then $\lim_{x\to 3} f(x) = 5$. **True** **False**

yes, in case where $x\to 3$: $3 \downarrow a$, $5 \downarrow L$

Justification: for ~~all~~ any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{7}$

Then if $|x-3| < \delta$, then $|f(x)-5| \leq 7|x-3| < 7\delta = 7\frac{\varepsilon}{7} = \varepsilon$

Problem 4. (10 pts) Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = 1/x$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)\cdot x}}{h} =$$

$$= \lim_{h\to 0} \frac{x-x-h}{(x+h)\cdot x} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{(x+h)\cdot x} \cdot \frac{1}{\cancel{h}} = \lim_{h\to 0} -\frac{1}{(x+h)\cdot x} =$$

$$= \boxed{-\frac{1}{x^2}}$$

Problem 5. (12 pts) A stone is thrown straight up from the ground. Its position *s(t)* in feet above the ground after *t* seconds is given by $s(t) = 48t - 16t^2$.

(a) (2 pts) When does the stone land back on the ground? needed $s(t)=0$

$\Rightarrow 48t-16t^2=0$

$16t(3-t)=0$ so after 3 second the stone will reach the ground

positive

$\Rightarrow \cancel{t=0}$ and $t=3$

(b) (4 pts) Find the average velocity of the stone in the time interval [0,2] seconds.

$$V_{avr} = \frac{S(2)-S(0)}{2-0} = \frac{48\cdot 2-16\cdot 4-0}{2} = 32\ ft/sec$$

(b) (6 pts) Use a limit to find the instantaneous velocity of the stone at *t*=2 seconds.

$$\lim_{h\to 0}\frac{s(2+h)-s(2)}{h} = \lim_{h\to 0}\frac{48(2+h)-16\cdot(2+h)^2-48\cdot 2+16\cdot 4}{h} =$$

$$= \lim_{h\to 0}\frac{\cancel{48\cdot 2}+48h-16(4+4h+h^2)-\cancel{48\cdot 2}+16\cdot 4}{h} =$$

$$= \lim_{h\to 0}\frac{48h-\cancel{16\cdot 4}-4h-h^2+\cancel{16\cdot 4}}{h} = \lim_{h\to 0}\frac{\cancel{h}(48-4-h)}{\cancel{h}} = \boxed{44\ ft/sec}$$

Problem 6. (12 pts) (a) (2pts) Write the definition for a function *f(x)* to be continuous at *x=a*.

$$\underset{1.}{\lim_{x\to a^+} f(x)} = \underset{2.}{\lim_{x\to a} f(x)} = \underbrace{\lim_{x\to a} f(x) = f(a)}_{3.}$$

(b) (5pts) Use this definition to determine whether or not the following function is continuous at *x=0*.

$$f(x)=\begin{cases}\dfrac{x^2+3}{x^2+1} & ,if\ x\le 0\\[2mm] \dfrac{\sin(3x)}{x} & ,if\ x>0\end{cases}$$

$$\lim_{x\to 0^+}\frac{\sin(3x)}{x} = \lim_{x\to 0^+}\frac{\sin(3x)}{x}\cdot\frac{3}{3} = \lim_{x\to 0^+}\overset{1}{\left(\frac{\sin(3x)}{(3x)}\right)}3 = 3$$

$$\lim_{x\to 0^-}\frac{x^2+3}{x^2+1} = 3 \qquad f(0) = \frac{0^2+3}{0^2+1} = 3$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 3 \Rightarrow f(x) \text{ is continoue at } x=0$$

(c) (5pts) List all asymptotes, vertical or horizontal (if any), of the function *f(x)* from part (b). Justify your answer with limits.

horizontal asymptote: $\boxed{\begin{array}{l} y=1 \text{ (when } x\to-\infty) \\ y=0 \text{ (when } x\to+\infty)\end{array}}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{x^2+3}{x^2+1} \underset{\text{paplage rule}}{=} 1$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{\sin(3x)}{x} = 0$$

$$-1 \le \sin(3x) \le 1 \ / \div x$$

$$-\frac{1}{x} \le \frac{\sin(3x)}{x} \le \frac{1}{x}$$

$$\lim_{x\to+\infty} \searrow \quad \downarrow 0 \quad \swarrow \lim_{x\to+\infty}$$

by Squeeze theorem

vertical asymptote:

on domain $x \le 0$ the function is defined everywhere because $x^2+1 \neq 0$ for real numbers

on domain $x > 0$ $f(x) = \frac{\sin(3x)}{x}$ is not define only on $x=0$ but it is not in the domain, so it is define on $x>0$ as well.

Thus $f(x)$ does not have vertical asymptotes

Problem 7. (12 pts) (a) (6 pts) Use the Intermediate Value Theorem to show that the equation $f(x) = x^3 - 4x + 2 = 0$ has three distinct real roots. Explain thoroughly. $f(x)$ is ~~co~~ polynomial & continuous everywhere

$f(0) = 2 > 0$
$f(1) = 1 - 4 + 2 = -1 < 0$

$\Big\}$ because $f(x)$ ~~do~~ is continuous and changes signs on interval $x \in (0,1)$ than by IVT it has a rooth on this interval

$f(-1) = -1 + 4 + 2 = > 0$
$f(-2) = -\cancel{8} + \cancel{8} + 2 = 2 > 0$
$f(-3) = -27 + 12 + 2 < 0$
$f(2) = 8 - 8 + 2 = 2 > 0$
~~$f(3) = 27 - 12$~~

another interval is $(-2, -3)$ (same reasoning) and $(1, 2)$

(b) (6 pts) Use the method of bisection to approximate one of the roots of the equation $x^3 - 4x + 2 = 0$ to within 0.25. Explain thoroughly.

By IVT there is a rooth on interval $(0, 1)$. but it is of size 1. Split it on half:

$f(0) > 0$
$f(0.5) = \left(\frac{1}{2}\right)^3 - 4 \cdot \frac{1}{2} + 2 = \frac{1}{8} > 0$
$f(1) < 0$

Since $f(x)$ does not change sign on interval $(0, \frac{1}{2})$ than I ~~will~~ can reduce my interval to $(\frac{1}{2}, 1)$ where $f(x)$ changes signs. Again reduce this new interval by half.

$f(\frac{1}{2}) > 0$
$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^3 - 4 \cdot \frac{3}{4} + 2 < 0$
$f(1) < 0$

$f(x)$ changes sign on interval $(\frac{1}{2}, \frac{3}{4})$ ~~which~~ i.e. by IVT it has root there and it's of size ~~0.25~~ 0.25 ~~$(\frac{3}{4} - \frac{1}{2} ...)$~~ $\left(\left|\frac{1}{2} - \frac{3}{4}\right| = \frac{1}{4}\right)$

Problem 8. (10 pts) Choose ONE of the following. Only ONE will be graded.

(A) Use geometry to prove the inequality $\sin x \le x \le \tan x$ for any $\in \left[0, \frac{\pi}{2}\right)$.

See notes, is the one with the unit circle, and subcribed triangle & sector in the circle and their areas are compared

(B) Write the general (ε, δ) definition for

$\lim_{x \to a} f(x) = L$

For all $\varepsilon > 0$, there is $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$

~~(C)~~ and then use this definition to prove that $\lim_{x \to 10} \underbrace{7x-3}_{f(x)} = 67$

a, L

~~(D)~~ side work:

$|f(x) - L| = |7x-3-67| =$

$= |7x-70| = 7|x-10| < \varepsilon$

$|x-10| < \frac{\varepsilon}{7}$

$\Rightarrow \delta = \frac{\varepsilon}{7}$

proof: let $\delta = \frac{\varepsilon}{7}$

then $|x-10| < \delta = \frac{\varepsilon}{7}$

$7|x-10| < \varepsilon$

$|7x-70| < \varepsilon$

$|\underbrace{7x-3}-67| < \varepsilon$

$|f(x) - L| < \varepsilon$ $\square$