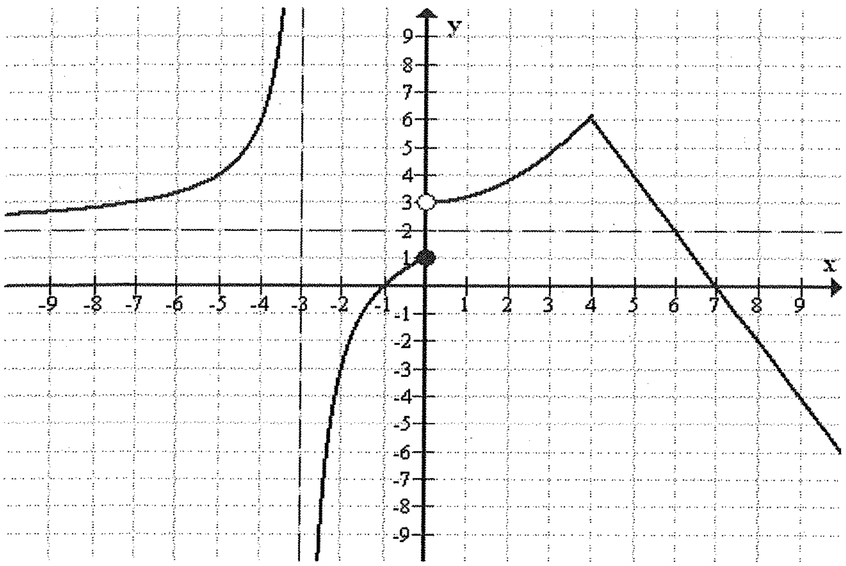
Name: Solution Key PID: ____________________

Spring 2017 -- MAC 2311- Exam 1 – Version B

There are 8 problems for a total of 110 points. **Show your work**; an answer alone, even correct, may get no credit. An illegible solution will not be graded. **Calculators are not allowed.**

Problem 1. (12pts) The graph of a function f is given below. Use the graph to answer the questions that follow.



(i) (7 pts) Find the following limits. You don't have to show any work for these, but specify if a limit is infinite or does not exist.

$\lim_{x\to -3^-} f(x) = +\infty$ $\lim_{x\to -3^+} f(x) = -\infty$ $\lim_{x\to -3} f(x) =$ D.N.E.

$\lim_{x\to 4} f(x) = 6$ $\lim_{x\to 0} f(x) =$ D.N.E.

$\lim_{x\to -\infty} f(x) = 2$ $\lim_{x\to +\infty} f(x) = -\infty$

(ii) (3 pts) Is f continuous everywhere? If not, give x value(s) at which f has a discontinuity. Specify if any of the discontinuities is removable.

$f(x)$ is not continuous at $x=-3$ and at $x=0$

Neither of these discontinuities are removable

(iii) (2 pts) Identify any point(s) x, where the function is continuous, but it is not differentiable. Specify if there is no such point x.

At $x=4$ the function is continuous, but it is not differentiable.

Problem 2. (30 pts) Find the following limits. Specify if a limit is infinite or does not exist. Show all work and explain clearly (5 pts each).

a) $\lim_{x \to 1^+} \frac{5x-1}{1-x} = \quad \frac{4}{0^-} = -\infty$

b) $\lim_{x \to -2} \frac{2x^2-8}{x^2-x-6} \overset{\frac{0}{0}}{=} \lim_{x \to -2} \frac{2(x^2-4)}{(x-3)(x+2)} = \lim_{x \to -2} \frac{2(x-2)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \frac{2 \cdot (-4)}{(-5)} = \boxed{\frac{8}{5}}$

c) $\lim_{x \to 6} \frac{\sqrt{2x+4}-4}{x-6} \overset{\frac{0}{0}}{=} \lim_{x \to 6} \frac{(\sqrt{2x+4}-4)(\sqrt{2x+4}+4)}{(x-6)(\sqrt{2x+4}+4)} =$

$= \lim_{x \to 6} \frac{2x+4-16}{(x-6)(\sqrt{2x+4}+4)} = \lim_{x \to 6} \frac{2\cancel{(x-6)}}{\cancel{(x-6)}(\sqrt{2x+4}+4)}$

$= \frac{2}{\sqrt{16}+4} = \frac{2}{8} = \boxed{\frac{1}{4}}$

d) $\lim_{x\to+\infty} \cos\left(\frac{2\pi x}{3x+1}\right) = -\frac{1}{2}$

Note that $\lim_{x\to+\infty} \frac{2\pi x}{3x+1} \underset{\substack{\uparrow \\ \text{Quick Rule}}}{=} \frac{2\pi}{3}$

Since cos is continuous

$$\lim_{x\to+\infty} \cos\left(\frac{2\pi x}{3x+1}\right) = \cos\left(\frac{2\pi}{3}\right) = \boxed{-\frac{1}{2}}$$

use the unit circle

e) $\lim_{x\to 0} \frac{x\sin(6x)}{\tan^2(3x)} =$

$$= \lim_{x\to 0} \frac{\cancel{x}\,\frac{\sin(6x)}{6x}\cdot 6\cancel{x}}{\frac{\tan^2(3x)}{(3x)^2}\cdot 9\cancel{x^2}} = \frac{6}{9} = \boxed{\frac{2}{3}}$$

f) $\lim_{t\to-\infty} \frac{\sqrt{3t^2-t+1}}{2t} = \lim_{t\to-\infty} \frac{\sqrt{t^2\left[3-\frac{1}{t}+\frac{1}{t^2}\right]}}{2t} = \lim_{t\to-\infty} \frac{|t|\sqrt{3-\frac{1}{t}+\frac{1}{t^2}}}{2t}$

$$\underset{\substack{\uparrow \\ t<0}}{=} \lim_{t\to-\infty} \frac{-\cancel{t}\cdot\sqrt{3-\frac{1}{t}+\frac{1}{t^2}}}{2\cancel{t}} = \boxed{-\frac{\sqrt{3}}{2}}$$

Problem 3. (12 pts) These are true or false questions. Answer (1pt) and give brief justification (2pts). Graph can serve as a justification.

(a) If a function *f* is continuous at *x=0*, then *f* is differentiable at *x=0*. **True** **False**

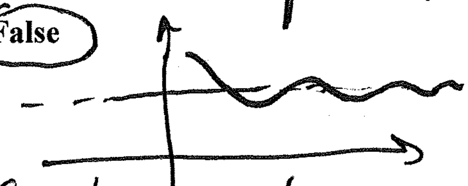
Justification: $f(x) = |x|$ is continuous at $x=0$ but it is not differentiable at $x=0$.

(b) If *f(x), g(x)* are continuous everywhere then $\frac{f(x)}{g(x)}$ is continuous everywhere. **True** **False**

Justification: $\frac{f(x)}{g(x)}$ is not continuous at points where $g(x)=0$. It will not be even defined at such points.

(c) A function can never cross its horizontal asymptote. **True** **False**

Justification:
$f(x) = \frac{\sin x}{x}$ has horiz. asymptote $y=0$, but graph crosses the x-axis



Graph example

(d) If a function satisfies $|f(x) -7| \le 5|x-3|$ for all real numbers x, then $\lim_{x\to 3} f(x) = 7$. **True** **False**

Justification: From the ε-δ definition, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{5}$

Then if $|x-3| < \delta \Rightarrow |f(x)-7| \le 5|x-3| < 5\delta = 5\cdot\frac{\varepsilon}{5} = \varepsilon$

Problem 4. (10 pts) Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = 1/x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{\frac{h}{1}} = \lim_{h\to 0} \frac{\overset{-1}{\cancel{x-x-h}}}{(x+h)\cdot x} \cdot \frac{1}{\cancel{h}},$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)\cdot x} = \boxed{-\frac{1}{x^2}}$$

Problem 5. (12 pts) A stone is thrown straight up from the ground. Its position $s(t)$ in feet above the ground after t seconds is given by $s(t) = 48t - 16t^2$.

(a) (2 pts) When does the stone land back on the ground?

$t = ?$ $s(t) = 0$

$0 = 48t - 16t^2 \iff 0 = 16t[3-t]$

$\cancel{t=0}$ or $\boxed{t = 3\,s}$

(b) (4 pts) Find the average velocity of the stone in the time interval [1,3] seconds.

$$V_{ave} = \frac{s(3) - s(1)}{3-1} = \frac{0 - (48\cdot 1 - 16\cdot 1^2)}{2} = -\frac{32}{2} = \boxed{-16\frac{ft}{s}}$$

$s(3) = 0$ (by (a))

$s(1) = 48\cdot 1 - 16\cdot 1^2 = 32$

(c)(6 pts) Use a limit to find the instantaneous velocity of the stone at t=1 second.

$$V_{inst.\ at\ t=1} = \lim_{h\to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \frac{48(1+h) - 16(1+h)^2 - 32}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{48} + 48h - \cancel{16} - 16\cdot 2h - 16h^2 - \cancel{32}}{h}$$

$$= \lim_{h\to 0} \frac{16h - 16h^2}{h} = \lim_{h\to 0} \frac{16\cancel{h}(1-h)}{\cancel{h}} = \boxed{16\frac{ft}{s}}$$

Problem 6. (12 pts) (a) (2pts) Write the definition for a function *f(x)* to be continuous at *x=a*.

$\lim_{x\to a} f(x) = f(a)$ (By this we mean that f is defined at $x=a$ and that the two sided limit $\lim_{x\to a} f(x)$ exist and is equal to $f(a)$.)

(b) (5pts) Use this definition to determine whether or not the following function is continuous at *x=0*.

$$f(x)=\begin{cases}\dfrac{2x^2+5}{x^2+1} & ,if\ x\le 0\\ \dfrac{\sin(3x)}{x} & ,if\ x>0\end{cases}$$

$f(0) = \frac{2\cdot 0^2+5}{0^2+1} = 5$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{2x^2+5}{x^2+1} = 5$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(3x)}{x} = 3$ $\#$

so $\lim_{x\to 0} f(x)$ D.N.E., so $f(x)$ is not continuous at $x=0$.

(b) (5pts) List all asymptotes, vertical or horizontal (if any), of the function *f(x)* from part (b). Justify your answer with limits.

The function has no vertical asymptotes (since x^2+1 is never 0 and $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ are finite)

The function does have horizontal asymptotes

$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{2x^2+5}{x^2+1} = 2 \Rightarrow \boxed{y=2 \text{ is a H.A.}}$ when $x\to-\infty$

$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{\sin(3x)}{x} = 0 \Rightarrow \boxed{y=0 \text{ is a H.A}}$ when $x\to+\infty$

Squeeze Thm $-\frac{1}{x} \le \frac{\sin(3x)}{x} \le \frac{1}{x}$

Problem 7. (12 pts) (a) (6 pts) Use the Intermediate Value Theorem to show that the equation $x^3 - 6x + 2 = 0$ has three distinct real roots. Explain thoroughly.

$f(x) = x^3 - 6x + 2$ is continuous everywhere since it is a polynomial

$f(0) = 2 > 0$, $f(1) = -3 < 0$ $\underset{\text{IVT}}{\Longrightarrow}$ there is $x_1 \in (0,1)$ so that $f(x_1) = 0$

$f(2) = -2 < 0$, $f(3) = 11 > 0$ $\Longrightarrow$ there is $x_2 \in (2,3)$ so that $f(x_2) = 0$

$f(-2) = 4 > 0$, $f(-3) = -7 < 0$ $\Longrightarrow$ there is $x_3 \in (-3,-2)$ so that $f(x_3) = 0$.

(b) (6 pts) Use the method of bisection to approximate one of the roots of the equation $x^3 - 6x + 2 = 0$ to within 0.25. Explain thoroughly.

Will apply the bisection method to the interval $(0,1)$ which we know contains a root of the equation

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6 \cdot \frac{1}{2} + 2 = \frac{1}{8} - 1 < 0$

since $f(0) = 2 > 0$ $\Longrightarrow x_1 \in \left(0, \frac{1}{2}\right)$

$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^3 - 6 \cdot \frac{1}{4} + 2 = \frac{1}{64} - \frac{3}{2} + 2 > 0$

since $f\left(\frac{1}{2}\right) < 0$ $\Longrightarrow x_1 \in \left(\frac{1}{4}, \frac{1}{2}\right)$

So one solution of the equation is in the interval

$(0.25, 0.5)$.

Problem 8. (10 pts) Choose ONE of the following. Only ONE will be graded.

(A) Use geometry to prove the inequality $\sin x \leq x \leq \tan x$ for any $x \in \left[0, \frac{\pi}{2}\right)$.

see notes or textbook

(B) Write the general (ε,δ) definition for
$\lim_{x \to a} f(x) = L$

for any given $\varepsilon > 0$, there is a $\delta > 0$ so that
if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

and then use this definition to prove that $\lim_{x \to 5}(20x - 3) = 97$

In this case

$$|f(x) - L| = |20x - 3 - 97| = 20|x-5|$$

So given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{20}$.

Then if $|x-5| < \delta \Rightarrow |f(x) - L| = 20|x-5| < 20\delta = 20 \cdot \frac{\varepsilon}{20} = \varepsilon$