NAME: Solution Key　　　　　　　　**Panther ID:** ____________

Exam 2 - MAC 2311　　　　　　　　Spring 2017

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (30 pts) Find dy/dx. Simplify when possible (6 pts each):

(a) $y = \frac{x^5}{5} - \frac{2}{\sqrt{x}} + 3\pi^4 = \frac{1}{5}x^5 - 2x^{-\frac{1}{2}} + \underbrace{3\pi^4}_{\text{constant}}$

$y' = \frac{1}{5}\cdot 5x^4 - 2\cdot\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} + 0$

$\boxed{y' = x^4 + \frac{1}{x^{\frac{3}{2}}} = x^4 + \frac{1}{x\sqrt{x}}}$

(b) $y = \sec(\tan(2x))$

Chain Rule

$y' = \sec(\tan(2x))\cdot\tan(\tan(2x))\cdot(\tan(2x))'$

$\boxed{y' = \sec(\tan(2x))\cdot\tan(\tan(2x))\cdot\sec^2(2x)\cdot 2}$

(c) $y = x^3e^{-2x}$

Product Rule

$y' = (x^3)'\cdot e^{-2x} + x^3(e^{-2x})'$

$y' = 3x^2e^{-2x} + x^3\cdot e^{-2x}\cdot(-2)$

$\boxed{y' = x^2e^{-2x}(3-2x)}$

Chain Rule

(d) $y = \arcsin(2^x)$

$$y' = (\arcsin(2^x))' \stackrel{\downarrow}{=} \frac{1}{\sqrt{1-(2^x)^2}} \cdot (2^x)'$$

$$\boxed{y' = \frac{1}{\sqrt{1-2^{2x}}} \cdot 2^x \cdot \ln 2} = \frac{2^x \cdot \ln 2}{\sqrt{1-2^{2x}}}$$

(e) $y = (x^2+1)^{\sin x}$

Logarithmic differentiation

$$\ln y = \ln\left((x^2+1)^{\sin x}\right) = \sin x \cdot \ln(x^2+1)$$

$$(\ln y)' = (\sin x \cdot \ln(x^2+1))'$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln(x^2+1) + \sin x \cdot \frac{2x}{x^2+1}$$

so $y' = (x^2+1)^{\sin x} \cdot \left[\cos x \cdot \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right]$

2. (10 pts) For the function $f(x)$ it is know that $f(1) = 2$ and $f'(1) = -5$.
Given that $g(x) = [f(x)]^3$, find

(a) (3 pts) $g(1)$

$$g(1) = [f(1)]^3 = 2^3 = 8$$

(b) (7 pts) $g'(1)$

Chain Rule

$$g'(x) = ([f(x)]^3)' \stackrel{\downarrow}{=} 3(f(x))^2 \cdot f'(x)$$

$$g'(1) = 3(f(1))^2 \cdot f'(1)$$

$$g'(1) = 3 \cdot 2^2 \cdot (-5) = -60$$

3. (12 points) These are True or False questions. Circle your answer AND give a brief justification.

(a) If $y = e^{kx}$, where k is a constant, then $dy/dx = ky$. **True** **False**

Justification: $\frac{dy}{dx} = (e^{kx})' = e^{kx} \cdot k = k \cdot y$

(b) If $g(x) = f(x)\sin x$ then $g'(x) = f'(x)\cos x$. **True** **False**

Justification: $g'(x) = f'(x)\sin x + f(x)\cos x$

Product Rule

(c) $\left((\tan x)^{-1}\right)' = \frac{1}{1+x^2}$ **True** **False**

Justification: $\left((\tan x)^{-1}\right)' = (-1)(\tan x)^{-2} \cdot \sec^2 x$

Note: $(\tan x)^{-1} \neq \tan^{-1} x$

$\frac{1}{\tan x}$ " " arctan x

(d) If $p(x)$ is a polynomial of degree 10, then its eleventh derivative, $p^{(11)}(x) = 0$. **True** **False**

Justification: Each derivative decreases the degree by 1.

$p^{(10)}(x) = \text{const}$ so $p^{(11)}(x) = (\text{const})' = 0$.

4. (12 pts) Show that $y = x\sin(5x)$ is a solution of a differential equation of the form $y'' + 25y = a\cos(5x)$, for a certain constant a that you should determine.

$y' = (x \cdot \sin(5x))' = 1 \cdot \sin(5x) + x\cos(5x) \cdot 5 = \sin(5x) + 5x\cos(5x)$

$y'' = 5\cos(5x) + 5 \cdot 1 \cdot \cos(5x) - 5x \cdot \sin(5x) \cdot 5$

$y'' = 10\cos(5x) - 25x\sin(5x)$

Thus $y'' = 10\cos(5x) - 25y$ so

$y'' + 25y = 10\cos(5x)$

So $y = x\sin(5x)$ is a solution of the above equation when $\underline{a = 10}$.

5. (12 pts) Find the equation of the tangent line to the curve $3x^3 - 2xy^2 = 2y^3 - 1$ at the point $(1,1)$.

Implicit diff.

$$(3x^3)' - (2xy^2)' = (2y^3)' - (1)'$$

$$9x^2 - 2y^2 - 2x \cdot 2y \cdot y' = 6y^2 \cdot y'$$

$$9x^2 - 2y^2 = 6y^2 y' + 4xy \cdot y' \Rightarrow$$

$$9x^2 - 2y^2 = (6y^2 + 4xy) \cdot y' \Rightarrow y' = \frac{9x^2 - 2y^2}{6y^2 + 4xy}$$

$$m_{tan} = \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{9-2}{6+4} = \frac{7}{10} \Rightarrow$$ eq. of tang. line is $\boxed{y - 1 = \frac{7}{10}(x-1)}$

6. (12 pts) Recall that the volume V of a circular cylinder having radius of the base r and height h is given by $V = \pi r^2 h$. Suppose we have a cylinder for which both dimensions r and h vary with time t.

(a) (6 pts) How are dV/dt, dr/dt and dh/dt related?

By product rule

$$\frac{dV}{dt} = \pi\left(2r \frac{dr}{dt} h + r^2 \cdot \frac{dh}{dt}\right)$$

(b) (6 pts) At a certain instant, the height is 8 cm and is increasing at 0.2 cm/s, while the radius is 10 cm and is decreasing at 0.1 cm/s. How fast is the volume changing at that instant? Give units to your answer. Is the volume increasing or decreasing at that instant?

$$h = 8\text{cm} \ \& \ \frac{dh}{dt} = 0.2 \frac{\text{cm}}{\text{s}}$$

$$r = 10\text{cm} \ \& \ \frac{dr}{dt} = -0.1 \frac{\text{cm}}{\text{s}}$$

$$\frac{dV}{dt} = \pi\left(2 \cdot 10 \cdot (-0.1) \cdot 8 + 10^2 \cdot (0.2)\right) = \pi(-16 + 20) = 4\pi \frac{\text{cm}^3}{\text{s}}$$

So, at that moment the volume is increasing at a rate of $4\pi \frac{\text{cm}^3}{\text{s}}$.

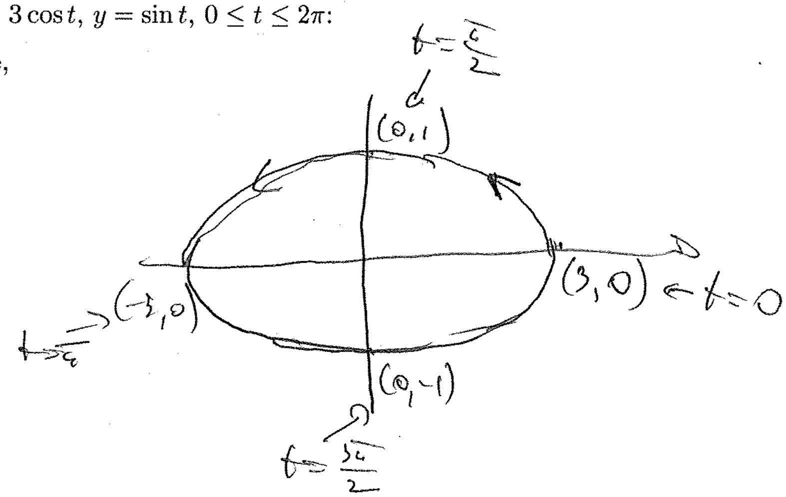
7. (12 pts) Given the parametric curve $x = 3\cos t$, $y = \sin t$, $0 \le t \le 2\pi$:

(a) (6 pts) Sketch the curve in the xy plane, marking the coordinates of axis intercepts and indicating orientation.

$\frac{x}{3} = \cos t$, $y = \sin t$

$\Downarrow$

$\left(\frac{x}{3}\right)^2 + y^2 = 1$ ← ellipse



(b) (6 pts) Find the coordinates of a point on the curve (if any) with the property that the tangent line to the curve at that point is parallel to the line $y = -\sqrt{3}x$.

We need to find a point P on the curve with

$$\frac{dy}{dx}\Big|_P = -\sqrt{3}$$

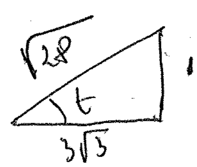
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-3\sin t} = -\frac{\cot t}{3}$$

So we want $-\frac{\cot t}{3} = -\sqrt{3} \iff \cot t = 3\sqrt{3} \iff$

$\iff \tan t = \frac{1}{3\sqrt{3}} \Rightarrow t = \arctan\left(\frac{1}{3\sqrt{3}}\right)$ ← not a nice angle (sorry!)

We need to find $x = 3\cos t$, $y = \sin t$ for t above.

Use the triangle method



so $\cos t = \frac{3\sqrt{3}}{\sqrt{28}}$, $\sin t = \frac{1}{\sqrt{28}}$

or $\left\{\cos t = -\frac{3\sqrt{3}}{\sqrt{28}}, \sin t = -\frac{1}{\sqrt{28}}\right.$ (if t in the 3rd quadrant)

Thus, there ~~points are~~ are two points with the required property:

$P_1\left(x_1 = \frac{9\sqrt{3}}{\sqrt{28}}, y_1 = \frac{1}{\sqrt{28}}\right)$; $P_2\left(x_2 = -\frac{9\sqrt{3}}{\sqrt{28}}, y_2 = -\frac{1}{\sqrt{28}}\right)$

8. (10 pts) Choose ONE:

(a) State and prove the quotient rule. (You can use product rule, chain rule, or logarithmic differentiation.)

(b) Find, with proof, the formula for $(\arccos x)'$.

(a) using logarithmic differentiation
(see your notes for other methods)

Let $y = \frac{f(x)}{g(x)}$

$$\ln y = \ln\left(\frac{f(x)}{g(x)}\right) = \ln(f(x)) - \ln(g(x)) \quad \Big|\ \text{Take } \frac{d}{dx}$$

$$(\ln y)' = (\ln(f(x)))' - (\ln(g(x)))'$$

$$\frac{1}{y}\cdot y' = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}$$

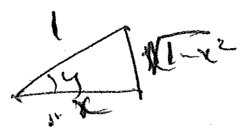
$$y' = y\cdot\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right) = \frac{f(x)}{g(x)}\cdot\left(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}\right)$$

So
$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)}{g(x)} - \frac{f(x)\cdot g'(x)}{(g(x))^2} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}}$$

(b) Let $y = \arccos x$. Then $\cos y = x$. Take $\frac{d}{dx}$ of both sides

$$-\sin y\cdot y' = 1 \Rightarrow y' = -\frac{1}{\sin y}$$

$\cos y = x = \frac{x}{1}$



so $\sin y = \frac{\sqrt{1-x^2}}{1}$

Thus $\boxed{(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}$

The proof based on the identity $\arcsin x + \arccos x = \frac{\pi}{2}$ is also acceptable.