NAME: Solution Key Panther ID: ____________

Exam 3 - MAC 2311 Spring 2017

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) Fill in the most appropriate words or symbols:

(a) A point x_0 is a critical point for the function $f(x)$, if $f'(x_0)$ is 0 or undefined.

(b) If $f''(x) < 0$, for all $x \in (a, b)$, then on the interval (a, b) the function f is concave down.

(c) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a relative minimum for the function $f(x)$.

(d) If $f(x)$ is differentiable on the interval $[a, b]$ and $f'(x) > 0$ for all $x \in [a, b]$, then on the interval $[a, b]$, f has an absolute maximum at $x =$ b.

2. (12 pts) (a) (8 pts) Find the local linear approximation of the function $f(x) = \cot x$ at $x_0 = \pi/4$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0) = \cot\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = (\cot x)' = -\csc^2 x$$

$$f'(x_0) = -\csc^2\left(\frac{\pi}{4}\right) = -\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} = -\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = -2$$

so $\boxed{\cot x \approx 1 - 2\left(x - \frac{\pi}{4}\right)}$

(b) (4 pts) Use part (a) to approximate $\cot 43°$ without using a calculator. (It's OK if your answer contains π.)

$180° \leftrightarrow \pi$ radians $\Rightarrow 43° = 43 \cdot \frac{\pi}{180} = \frac{43\pi}{180}$ radians

$$\cot(43°) = \cot\left(\frac{43\pi}{180}\right) \approx 1 - 2\left(\frac{43\pi}{180} - \frac{45\pi}{4 \cdot 45}\right)$$

so $\cot(43°) \approx 1 + \frac{4\pi}{180}$ or $\boxed{\cot(43°) \approx 1 + \frac{\pi}{45}}$

3. (16 pts) Compute each of the following limits:

(a) (6 pts) $\lim_{x\to 0} \dfrac{e^{5x}-1}{\sin(3x)}$ $\overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \dfrac{(e^{5x}-1)'}{(\sin(3x))'} = \lim_{x\to 0} \dfrac{5\cdot e^{5x}}{\cos(3x)\cdot 3} = \boxed{\dfrac{5}{3}}$

(b) (10 pts) $\lim_{x\to+\infty} \left(1-\frac{3}{x}\right)^x$ $\overset{1^\infty}{=} \lim_{x\to+\infty} e^{\ln\left[\left(1-\frac{3}{x}\right)^x\right]} = \lim_{x\to+\infty} e^{x\cdot\ln\left(1-\frac{3}{x}\right)}$

$= e^{\lim_{x\to+\infty} x\cdot\ln\left(1-\frac{3}{x}\right)}$

$\lim_{x\to+\infty} x\cdot\ln\left(1-\frac{3}{x}\right) \overset{\infty\cdot 0}{=} \lim_{x\to+\infty} \dfrac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{l'H}{=}}$

$= \lim_{x\to+\infty} \dfrac{\frac{1}{1-\frac{3}{x}}\cdot(-3)\cdot(-1)x^{-2}}{(-1)x^{-2}} = -3$

so $\boxed{\lim_{x\to+\infty}\left(1-\frac{3}{x}\right)^x = e^{-3} = \frac{1}{e^3}}$

4. (24 pts) Find the indicated antiderivatives (6 pts each):

(a) $\int \left(\frac{3}{2} - 3^x + \frac{1}{\sqrt{1-x^2}}\right) dx$ $= \frac{3}{2}x - \frac{3^x}{\ln 3} + \arcsin x + c$

(b) $\int x \sec^2(x^2)\, dx$ $=$ $\int \sec^2(w) \cdot \frac{1}{2} dw = \frac{1}{2}\tan w + c$

sub. $w = x^2$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$= \frac{1}{2}\tan(x^2) + c$

(c) $\int \sin^3(3x)\cos(3x)\, dx$ $=$ $\int w^3 \cdot \frac{1}{3} dw = \frac{1}{3}\cdot\frac{1}{4}w^4 + c$

sub. $w = \sin(3x)$

$dw = \cos(3x)\cdot 3\,dx$

$\frac{1}{3}dw = \cos(3x)\,dx$

$= \frac{1}{12}\sin^4(3x) + c$

(d) $\int \frac{t}{t^4+1}\, dt$ $=$ $\int \frac{\frac{1}{2}du}{u^2+1} = \frac{1}{2}\arctan(u) + c$

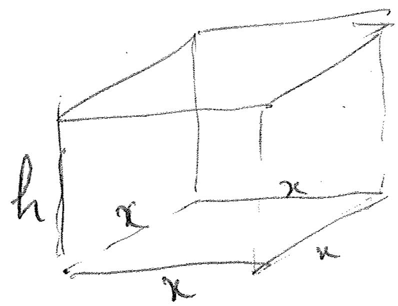
sub. $u = t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$= \frac{1}{2}\arctan(t^2) + c$

5. (12 pts) If 2400 cm^2 of cardboard material is available to make a closed box with a square base, find the largest possible volume of the box. (Hint: You are given the surface area of the box and you need to maximize its volume.)



Let x = length of side for the square base
h = height of the box
S = surface area of the box
V = volume of the box

$$V = x^2 \cdot h$$

$$S = \underbrace{2x^2}_{\text{top \& bottom}} + \underbrace{4xh}_{\text{the 4 sides}} = 2400 \quad \leftarrow \text{given}$$

~~Given~~ We used the above relation to solve for h

$$4xh = 2400 - 2x^2 \Rightarrow \boxed{h = \frac{2400}{4x} - \frac{2x^2}{4x} = \frac{600}{x} - \frac{x}{2}}$$

Replace h in the volume formula

$$V(x) = x^2 \cdot \left(\frac{600}{x} - \frac{x}{2}\right) = 600x - \frac{x^3}{2}$$

Want to maximize $V(x)$ if $0 < x < \sqrt{1200}$ ← $x^2 < 1200$ ← $2x^2 + 4xh = 2400 \Rightarrow 2x^2 < 2400$

$$V'(x) = 600 - \frac{3x^2}{2}$$

$$V'(x) = 0 \Leftrightarrow 600 = \frac{3x^2}{2} \Leftrightarrow x^2 = 400 \Leftrightarrow \boxed{x = 20} \text{ critical point}$$

Since $V''(x) = -3x < 0$ (as $x > 0$) $\Rightarrow V(x)$ is concave down on the domain

Thus, the critical pt $x = 20$ is an absolute maximum for $V(x)$

Maximum volume is $V(20) = 600 \cdot 20 - \frac{(20)^3}{2} = 12000 - 4000 = \boxed{8000 \text{ cm}^3}$

6. (20 + 2 pts) The steps of this problem should lead you to a complete graph of the function $f(x) = x^2e^{-x}$.

(a) (1 pts) The domain of this function is <u>all reals</u>

(b) (3 pts) Find the derivative $f'(x)$ and write it in factored form.

$$f'(x) = (x^2)'e^{-x} + x^2(e^{-x})' = 2xe^{-x} + x^2e^{-x}\cdot(-1)$$

$$\text{so } \boxed{f'(x) = xe^{-x}(2-x)}$$

(c) (3 pts) Find the critical points of f.

$$f'(x) = 0 \iff \boxed{x=0} \text{ or } \boxed{x=2}$$

← critical points

(d) (4 pts) Do a sign chart for f' and specify the intervals where f is increasing, respectively decreasing.

x	$-\infty$		0		2		$+\infty$
$f'(x)$		$-\ -\ -\ -\ -$	0	$+\ +\ +$	0	$-\ -\ -\ -$	
$f(x)$	$+\infty$	↘	0	↗	$4e^{-2}$	↘	0

$f(x)$ is decreasing on $(-\infty, 0)$ and $(2, +\infty)$

$f(x)$ is increasing on $(0, 2)$

(e) (4 pts) Determine the end behavior of the function $f(x) = x^2e^{-x}$ and any eventual horizontal asymptotes.

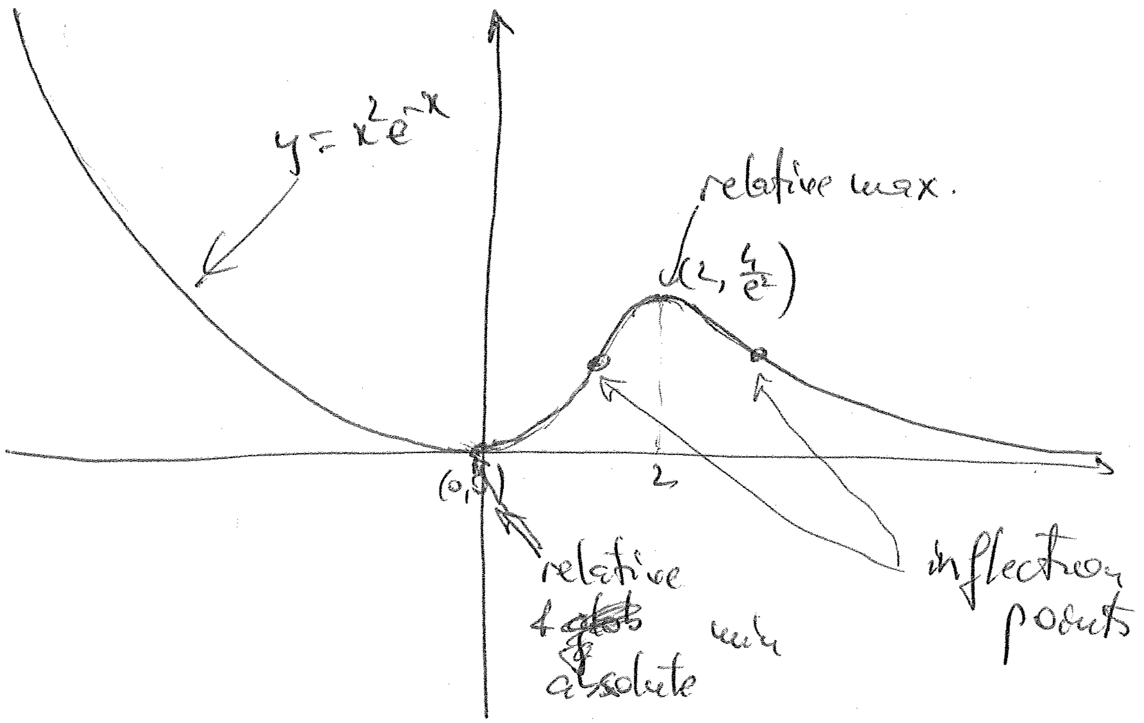
$$\lim_{x\to+\infty} x^2e^{-x} \overset{\infty\cdot 0}{=} \lim_{x\to+\infty} \frac{x^2}{e^x} \underset{\text{l'H twice}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty} \frac{2x}{e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = \boxed{0}$$

so $y=0$ is a H.A when $x \to +\infty$

$$\lim_{x\to-\infty} x^2e^{-x} = (+\infty)\cdot(+\infty) = +\infty$$

so no horiz. asymptote when $x \to +\infty$

(f) (5 pts) Using all the previous steps, sketch the graph of $f(x) = x^2e^{-x}$. Label on the graph the coordinates of critical points and specify their type (relative/absolute min/max).



Bonus 2 pts: I did not ask you to do the analysis of the second derivative. Without computing the second derivative, how many inflection points do you expect?

One expects two inflection points (but to be sure of this one should do the sign chart for the second derivative)

7. (12 pts) On the moon the acceleration due to gravity is $g = -5$ ft/sec^2. An astronaut jumps up from the surface of the moon with an initial upward velocity of 10 ft/sec.

(a) (6 pts) Find the formulas for the velocity $v(t)$ and the height above the ground $s(t)$ of the astronaut after t seconds since the start of its jump.

(b) (3 pts) How high does the astronaut go?

(c) (3 pts) How long does it take to get back on the ground?

(a) $v(t) = at + v_0$

$s(t) = \frac{at^2}{2} + v_0 t + s_0$ } equations of motion when acceleration is constant

For our case,

$$v(t) = -5t + 10$$
$$s(t) = -\frac{5t^2}{2} + 10t$$

(as $a = g = -5$, $v_0 = 10$, $s_0 = 0$)

(b) At the maximum height, $v(t) = s'(t) = 0$

So $0 = -5t + 10 \Rightarrow t = 2s$,

So after 2 s max. height is achieved

$h_{max} = s(2) = -\frac{5 \cdot 2^2}{2} + 10 \cdot 2 = 10$ ft

(c) $t = ?$ for $s(t) = 0$

$0 = -\frac{5t^2}{2} + 10t$ or $0 = t\left(-\frac{5t}{2} + 10\right)$

$t = 0$ or $t = 4$ s

So it takes 4 seconds for astronaut to get back on ground.