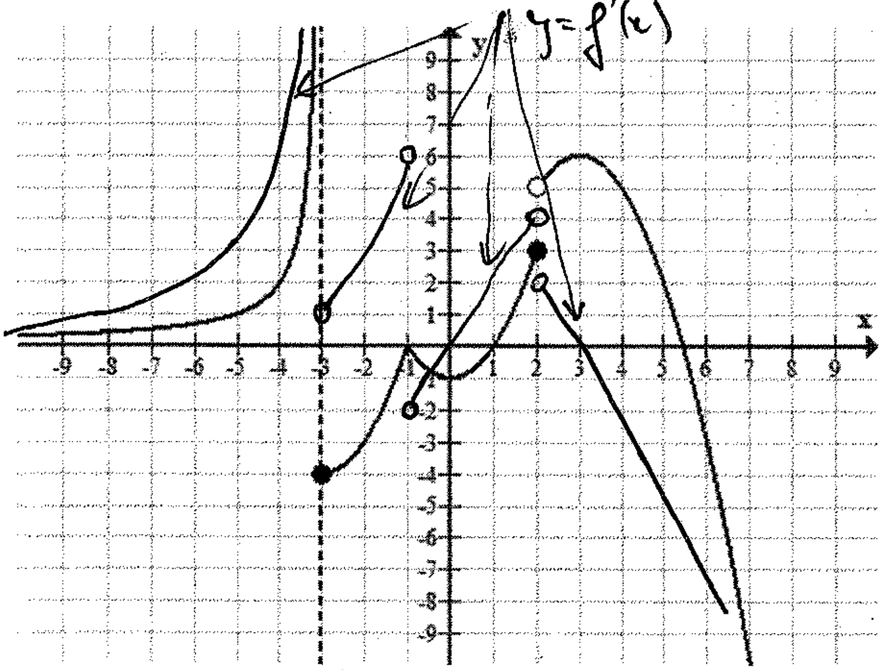
Name: Solution Key PID: ______________________

Summer 2019 -- MAC 2311- Exam 1

There are 8 problems for a total of 109 points. **Show your work**; an answer alone, even correct, may get no credit. An illegible solution will not be graded. **Calculators are not allowed**.

Problem 1. (17 pts) The graph of a function f is given below. Use the graph to answer the questions that follow.



(i) (7 pts) Find the following limits (you don't have to show any work here)

$\lim_{x\to -3^-} f(x) = +\infty$ $\lim_{x\to -3^+} f(x) = -4$ $\lim_{x\to -3} f(x) =$ D.N.E.

$\lim_{x\to 2} f(x) =$ D.N.E. $\lim_{x\to 0} f(x) = -1$

$\lim_{x\to -\infty} f(x) = 0$ $\lim_{x\to +\infty} f(x) = -\infty$

(ii) (2 pts) Specify the domain of the function f.

all real numbers

(iii) (3 pts) Is f continuous everywhere? If not, give x value(s) at which f has a discontinuity. Specify if any of the discontinuities is removable.

No, f is not continuous at $x=-3$ and at $x=2$

Neither one of these is a removable discontinuity.

(iv) (2 pts) Identify any point(s) x, where the function is not differentiable. Specify if there is no such point x.

f is not differentiable at $x=-3$ and at $x=2$ (as is not continuous at those points) and at $x=-1$ (as it has a corner point there)

(v) (3pts) On the same coordinate system, sketch the graph of $f'(x)$.

see the sketched graph above

Problem 2. (30 pts) Find the following limits. If a limit is infinite or does not exist, specify so. (5 pts each).

a) $\lim\limits_{x\to -3^+} \frac{x}{3+x}$ $= \frac{-3}{0^+} = -\infty$

b) $\lim\limits_{x\to 3} \frac{x^2-x-6}{9x-x^3} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{x\underbrace{\cancel{(3-x)}}_{(-1)}(3+x)} = \frac{5}{3(-1)\cdot 6} = \boxed{-\frac{5}{18}}$

c) $\lim\limits_{x\to +\infty} \frac{x^2-x-6}{9x-x^3} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to +\infty} \frac{x^2}{-x^3} = \lim\limits_{x\to +\infty} \frac{1}{-x} = \boxed{0}$

Junk Rule

More rigorously (but the above receives full credit):

$$\lim_{x\to +\infty} \frac{x^2\left(1-\frac{1}{x}-\frac{6}{x^2}\right)}{x^3\left(\frac{9}{x^2}-1\right)} = \lim_{x\to +\infty} \frac{\left(1-\frac{1}{x}-\frac{6}{x^2}\right)}{x\left(\frac{9}{x^2}-1\right)} = \frac{1}{\infty\cdot(-1)} = \boxed{0}$$

d) $\lim\limits_{x\to +\infty} \cos(x) =$ D.N.E. because $\cos x$ keeps oscillating between -1 and 1 as $x \to +\infty$ (the y-value does not approach any particular value as $x\to+\infty$)

e) $\lim\limits_{x\to 0}\frac{\tan 2x}{\sin 5x}$ $\overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\frac{\tan(2x)}{2x}\cdot 2\cancel{x}}{\frac{\sin(5x)}{5x}\cdot 5\cancel{x}} = \frac{2}{5}$ $\left(\text{we use } \lim\limits_{\theta\to 0}\frac{\tan\theta}{\theta}=1 \quad \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right)$

f) $\lim\limits_{x\to+\infty}\sqrt{x^2+3x} - x$ $\underset{\text{multiply by conjugate}}{\overset{\infty-\infty}{=}} \lim\limits_{x\to+\infty}\dfrac{\left(\sqrt{x^2+3x}-x\right)\left(\sqrt{x^2+3x}+x\right)}{\sqrt{x^2+3x}+x} =$

$= \lim\limits_{x\to+\infty}\dfrac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to+\infty}\dfrac{3x}{\sqrt{x^2\left(1+\frac{3}{x}\right)}+x} =$

$= \lim\limits_{x\to+\infty}\dfrac{3x}{x\sqrt{1+\frac{3}{x}}+x} = \lim\limits_{x\to+\infty}\dfrac{3\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{3}{x}}+1\right)} = \boxed{\frac{3}{2}}$

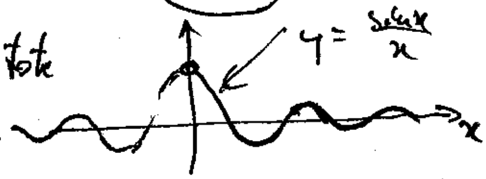
Also gives full credit the quick application of Junk Rule after the first step, that is:

$\lim\limits_{x\to+\infty}\dfrac{3x}{\sqrt{x^2+3x}+x} \underset{\text{Junk Rule}}{=} \lim\limits_{x\to+\infty}\dfrac{3x}{\sqrt{x^2}+x} = \lim\limits_{x\to+\infty}\dfrac{3x}{x+x} = \boxed{\frac{3}{2}}$

Problem 3. (9 pts) These are true or false questions. Answer (1pt) and give brief justification (2pts). Graph can serve as a justification.

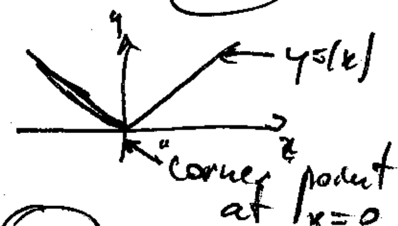
(a) The graph of a function can never cross its horizontal asymptote. **True** **False**

Justification: $f(x) = \frac{\sin x}{x}$ has horizontal asymptote $y=0$ (as $\lim_{x \to \pm\infty} \frac{\sin x}{x} = 0$), but the graph crosses this asymptote infinitely many times



(b) If a function f is continuous at $x=0$, then f is differentiable at $x=0$. **True** **False**

Justification: $f(x) = |x|$ is continuous at $x=0$ but it is not differentiable at $x=0$



(c) The equation $x^3 = 3x^2 + 1$ has a solution in the interval [3,4]. **True** **False**

Justification: Equation is equivalent to $x^3 - 3x^2 - 1 = 0$

$f(x) = x^3 - 3x^2 - 1$ is a polynomial, hence is continuous everywhere

$f(3) = 3^3 - 3\cdot 3^2 - 1 = -1 < 0$ $\Rightarrow$ By I.V.T. there is a point $x^* \in [3,4]$

$f(4) = 4^3 - 3\cdot 4^2 - 1 = 15 > 0$ so that $f(x^*) = 0$

Problem 4. (8 pts) Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$

multiply by conjugate

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x}) \cdot h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h}^{1} - \cancel{x}}{(\sqrt{x+h} + \sqrt{x}) \cdot \cancel{h}} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

Problem 5. (15 pts) Compute *f'(x)* for the following. You don't need to use limits here. (5 points each)

a) $f(x) = x^3e^x$

$$f'(x) = (x^3e^x)' \overset{\text{Product Rule}}{=} 3x^2e^x + x^3e^x = x^2e^x(3+x)$$

(either form is acceptable, but in the future we will often need the derivative in factored form

b) $f(x) = \frac{x^2-4}{3x+1}$

$$f'(x) = \left(\frac{x^2-4}{3x+1}\right)' \overset{\text{Quotient Rule}}{=} \frac{2x(3x+1) - (x^2-4)\cdot 3}{(3x+1)^2} = \frac{6x^2+2x-3x^2+12}{(3x+1)^2}$$

$$f'(x) = \frac{3x^2+2x+12}{(3x+1)^2}$$

c) $f(x) = \pi^2 + 3\sqrt{x} - \frac{5}{x}$

$$f(x) = \underbrace{\pi^2}_{\text{constant}} + 3x^{\frac{1}{2}} - 5x^{-1}$$

Power Rule for these

so $f'(x) = (\pi^2)' + (3x^{\frac{1}{2}})' - (5x^{-1})' =$

$$= 0 + 3\cdot\frac{1}{2}x^{-\frac{1}{2}} - 5\cdot(-1)x^{-2}$$

Thus $f'(x) = \frac{3}{2}x^{-\frac{1}{2}} + 5x^{-2} = \boxed{\frac{3}{2\sqrt{x}} + \frac{5}{x^2}}$

either form is ok.

Problem 6. (12 pts) (a) (2pts) Write the limit definition of continuity for a function *f(x)* at *x=a*.

$$\lim_{x\to a} f(x) = f(a)$$

(b) (5pts) Use this definition to determine whether or not the following function is continuous at *x=0*.

$$f(x) = \begin{cases} \dfrac{x^2+5}{2x^2+1} & , if\ x \le 0 \\ \dfrac{sin(3x)}{x} & , if\ x > 0 \end{cases}$$

$$f(0) = \frac{0^2+5}{2\cdot 0^2+1} = \boxed{5} \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2+5}{2x^2+1} = \boxed{5}$$

$$\text{but } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin(3x)}{x} = 3$$

so, no, the function is not continuous at $x=0$, as the two-sided limit $\lim_{x\to 0} f(x)$ does not exist.

(c)(5pts) List all asymptotes, vertical or horizontal (if any), of the function *f(x)* from part (b). Justify your answer with limits.

The function does have two horizontal asymptotes.

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x^2+5}{2x^2+1} \underset{\text{Leah Rule}}{=} \frac{1}{2}$$, so $y=\frac{1}{2}$ is a H.A. when $x\to -\infty$

$$\lim_{x\to +\infty} \frac{\sin(3x)}{x} = 0$$ (by Squeeze Theorem $-1 \le \sin(3x) \le 1$), so $y=0$

so $-\frac{1}{x} \le \frac{\sin(3x)}{x} \le \frac{1}{x}$

$x\to +\infty$: $\to 0$

There are no vertical asymptotes

(Note that $x=0$ is not a V.A. as none of the one sided limits $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ is $+\infty$ or $-\infty$)

Problem 7. (8 pts) A particle is moving on the x-axis. Its position (in cm) at time t (in seconds) is given by $x(t)=t^3+t-9$. Answer the following (2 pts each)

a) Find the instantaneous velocity at time t=2 s.

$$v(t) = \frac{dx}{dt} = 3t^2+1$$

$$v(2) = 3\cdot 2^2+1 = 13 \ \text{cm}/\text{s}$$

b) Find the acceleration at t=2 s.

$$a(t) = \frac{dv}{dt} = 6t$$

$$a(2) = 6\cdot 2 = 12 \ \text{cm}/\text{s}^2$$

c) Find the average velocity over the time interval [0,1].

$$\underset{\text{for } t\in[0,1]}{V_{ave}} = \frac{\Delta x}{\Delta t} = \frac{x(1)-x(0)}{1-0} = \frac{(1+1-9)-(-9)}{1} = \frac{2\cancel{-9}\cancel{+9}}{1} = 2\ \frac{\text{cm}}{\text{s}}$$

d) Find the time interval when the particle is moving to the left.

This corresponds to finding the time when the instantaneous velocity $v(t)$ is negative.

But $v(t) = 3t^2+1$ is always positive

Thus this particle is always moving to the right!

Problem 8. (10 pts) Choose ONE:

(A) State and prove the formula for the derivative of a product (the product rule).

(B) Prove the power rule for the case of a positive integer power.

See textbook or class notes