NAME: Solution Key Panther ID: ____________

Exam 2 - MAC 2281 Spring 2019

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (40 pts) In each case, find the indicated derivative. Simplify your answer when possible (8 pts each).

(a) If $y = 3\arctan x - \dfrac{2}{\tan x}$, find dy/dx .

$$y = 3\arctan x - 2\cot x$$

$$\boxed{\frac{dy}{dx} = \frac{3}{1+x^2} + 2\csc^2 x}$$

Note: Applying quotient rule (correctly) for $\left(\frac{2}{\tan x}\right)'$ is also OK, but you'd have to work harder

(b) If $y = \sqrt{1+\cos(t^2)}$, find dy/dt .

$$y = \left(1+\cos(t^2)\right)^{\frac{1}{2}}$$

$$\frac{dy}{dt} = \frac{1}{2}\left(1+\cos(t^2)\right)^{-\frac{1}{2}} \cdot \left(1+\cos(t^2)\right)' = \frac{1}{\cancel{2}}\left(1+\cos(t^2)\right)^{-\frac{1}{2}} \cdot \left(-\sin(t^2)\cdot \cancel{2}t\right)$$

$$\boxed{\frac{dy}{dt} = \frac{-t\sin(t^2)}{\sqrt{1+\cos(t^2)}}}$$

(c) If $y = e^{\sin t}$, find d^2y/dt^2 .

$$\frac{dy}{dt} = \left(e^{\sin t}\right)' = e^{\sin t}\cdot\cos t = (\cos t)\cdot e^{\sin t}$$

$$\frac{d^2y}{dt^2} = \left(\cos t \cdot e^{\sin t}\right)' \overset{\text{product rule}}{=} -\sin t \cdot e^{\sin t} + \cos t \cdot e^{\sin t}\cdot \cos t$$

$$\boxed{\frac{d^2y}{dt^2} = e^{\sin t}\left(\cos^2 t - \sin t\right)}$$

(d) If $y = \sec^2(3x+1)$, find dy/dx .

$$y = (\sec(3x+1))^2$$

$$\frac{dy}{dx} = 2\sec(3x+1)\cdot(\sec(3x+1))' = 2\sec(3x+1)\cdot\sec(3x+1)\tan(3x+1)\cdot 3$$

$$\boxed{\frac{dy}{dx} = 6\sec^2(3x+1)\tan(3x+1)}$$

(e) If $y = (\ln x)^x$, find dy/dx . Hint: Apply logarithmic differentiation.

$$\ln y = \ln((\ln x)^x)$$

$$\ln y = x\cdot\ln(\ln x) \quad | \text{Take } \frac{d}{dx}$$

$$(\ln y)' = (x\cdot\ln(\ln x))'$$

$$\frac{1}{y}\cdot y' = 1\cdot\ln(\ln x) + \cancel{x}\cdot\frac{1}{\ln x}\cdot\frac{1}{\cancel{x}}$$

$$\text{so } \boxed{\frac{dy}{dx} = (\ln x)^x\cdot\left[\ln(\ln x) + \frac{1}{\ln x}\right]}$$

2. (10 pts) The position of a particle moving along a coordinate line is $s = \sqrt{1+4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ seconds. Write the units for your answers.

$$s(t) = (1+4t)^{\frac{1}{2}} \quad m$$

$$v(t) = \frac{ds}{dt} = \frac{1}{\cancel{2}}(1+4t)^{-\frac{1}{2}}\cdot \overset{2}{\cancel{4}} = \frac{2}{\sqrt{1+4t}} \quad \frac{m}{s}$$

$$a(t) = \frac{dv}{dt} = \left(2(1+4t)^{-\frac{1}{2}}\right)' = \cancel{2}\cdot\left(-\frac{1}{\cancel{2}}\right)(1+4t)^{-\frac{3}{2}}\cdot 4 = -\frac{4}{(\sqrt{1+4t})^3} \quad \frac{m}{s^2}$$

$$\text{so} \quad \boxed{v(6) = \frac{2}{\sqrt{1+4\cdot 6}} = \frac{2}{\sqrt{25}} = \frac{2}{5} \quad \frac{m}{s}}$$

$$\boxed{a(6) = -\frac{4}{(\sqrt{1+4\cdot 6})^3} = -\frac{4}{5^3} = -\frac{4}{125} \quad \frac{m}{s^2}}$$

3. (20 points) These are True or False questions. Circle your answer AND give a brief justification.

(a) If $h(x) = \tan(g(x))$, then $h'(x) = \sec^2(g(x)) + \tan(g'(x))$. **True** **False**

Justification: Chain Rule should be applied

$h'(x) = (\tan(g(x)))' = \sec^2(g(x)) \cdot g'(x)$

(b) If $y = \sin(5x)$ then $y'' + 25y = 0$. **True** **False**

Justification: $y' = 5\cos(5x)$, $y'' = -25\sin(5x) = -25y$

so $y'' + 25y = 0$

(c) $\arcsin(\pi/2) = 1$ **True** **False**

Justification: Domain of arcsin is $[-1,1]$ and $\frac{\pi}{2} > 1$, so $\arcsin\left(\frac{\pi}{2}\right)$ does not even make sense.

(In other words, there is no angle θ so that $\sin\theta = \frac{\pi}{2}$, as $\frac{\pi}{2} > 1$)

(d) The function $f(x) = x^3 + 9x - 5$ is one-to-one over its domain $(-\infty, \infty)$. **True** **False**

Justification: $f'(x) = 3x^2 + 9 > 0$. Thus $f(x)$ is strictly increasing on $(-\infty, +\infty)$, so it is one-to-one on $(-\infty, +\infty)$.

(e) If $g(x) = f(x^3)$, then $g'(1) = 3f'(1)$. **True** **False**

Justification:

$g'(x) = f'(x^3) \cdot 3x^2$ by Chain Rule

$g'(1) = f'(1) \cdot 3$

4. (10 pts) Use implicit differentiation to find the slope of the tangent line to the curve $x^2 + y^2 - xy = 7$ at $(x, y) = (-1, 2)$.

We need to find $\left.\frac{dy}{dx}\right|_{(-1,2)}$

First we find $\frac{dy}{dx}$ by differentiating both sides of $x^2 + y^2 - xy = 7$

$$2x + 2y \cdot y' - (1 \cdot y + x \cdot y') = 0$$

$$2x - y + 2y \cdot y' - xy' = 0$$

$$(2y - x)y' = y - 2x$$

so $\boxed{y' = \frac{dy}{dx} = \frac{y - 2x}{2y - x}}$

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{2 - 2\cdot(-1)}{2\cdot 2 - (-1)} = \boxed{\frac{4}{5}}$$

Thus, the slope of the tangent line to the curve at $(-1,2)$ is $\boxed{m = \frac{4}{5}}$

5. (10 pts) (a) (6 pts) Find the local linear approximation of $f(x) = \sqrt[3]{x}$ at $x_0 = 8$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \quad \text{(General local lin. approx. formula)}$$

$$f(x_0) = \sqrt[3]{8} = 2$$

$$f'(x) = (\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2}$$

$$f'(x_0) = f'(8) = \frac{1}{3(\sqrt[3]{8})^2} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

so $\boxed{\sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8)}$

(b) (4 pts) Use part (a) to estimate $\sqrt[3]{7.97}$.

$$\sqrt[3]{7.97} \approx 2 + \frac{1}{12}(7.97 - 8)$$

$$\sqrt[3]{7.97} \approx 2 + \frac{1}{12}(-0.03) \qquad \text{or } \sqrt[3]{7.97} \approx 1.9975$$

$$\text{or } \sqrt[3]{7.97} \approx 2 - \frac{0.01}{4} \qquad \text{or } \sqrt[3]{7.97} \approx 2 - 0.0025$$

6. (10 pts) The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit are related by $V = IR$. Suppose that V, I, R all depend on time. At a certain moment, the voltage V is 12 volts and is increasing at the rate 0.5 volts/second, while the current I is 3 amps and decreasing at a rate of 0.2 amps/second. Find the rate at which R is changing at that moment. Is R increasing or decreasing?

Take $\frac{d}{dt}$ in both sides of the relation $V = I \cdot R$

$$\boxed{\frac{dV}{dt} = \frac{dI}{dt} \cdot R + I \cdot \frac{dR}{dt}} \quad (*)$$

This is the relation between the rates

We are given that a certain moment

$$V = 12 \text{ volts}, \quad \frac{dV}{dt} = 0.5 \frac{\text{volts}}{s}, \quad I = 3 \text{ amps}, \quad \frac{dI}{dt} = -0.2 \frac{\text{amps}}{s}$$

We have to find $\frac{dR}{dt}$ at that moment

From $V = I \cdot R$ we get that $R = \frac{V}{I} = \frac{12}{3} = 4$ ohms (at that moment) and we substitute all the given data in $(*)$ and solve for $\frac{dR}{dt}$.

$$0.5 = (-0.2) \cdot 4 + 3 \cdot \frac{dR}{dt}$$

$$0.5 = -0.8 + 3\frac{dR}{dt} \Rightarrow \boxed{\frac{dR}{dt} = \frac{1.3}{3} \frac{\text{ohms}}{s}} \approx 0.433 \frac{\text{ohms}}{s}$$

Thus, R is increasing at that moment at a rate of $0.4\overline{3} \frac{\text{ohms}}{s}$

7. (10 pts) Choose ONE:

(a) Use logarithmic differentiation to prove the general power rule for derivatives.

(b) Find, with proof, the formula for $(\arcsin x)'$.

(a) Let $y = x^p$, where $p \in \mathbb{R}$

$\ln y = \ln(x^p) = p \cdot \ln x$ | Take $\frac{d}{dx}$ of both sides

$(\ln y)' = (p \cdot \ln x)'$

$\frac{1}{y} \cdot y' = p \cdot \frac{1}{x}$ so $y' = p \cdot \frac{y}{x}$

or $(x^p)' = p \cdot \frac{x^p}{x} = p \cdot x^{p-1}$

(b) Let $y = \arcsin x$. This is equivalent with

$\sin y = x$. Take $\frac{d}{dx}$ of both sides

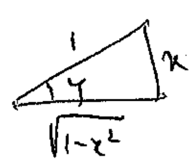
$(\sin y)' = (x)'$

$\cos y \cdot y' = 1$ so $y' = \frac{1}{\cos y}$

but if $\sin y = x$, $\cos y = \sqrt{1 - x^2}$

(either by using the identity $\sin^2 y + \cos^2 y = 1$

or by the triangle method



$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

Thus $\boxed{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$