NAME: Solution Key Panther ID: ____________

Exam 3 - MAC 2281 Spring 2019

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (8 pts) Fill in the most appropriate words or symbols:

(a) If x_0 is a critical point for the function $f(x)$, then $f'(x_0)$ is zero or undefined.

(b) If $f''(x) < 0$, for all $x \in (a, b)$, then on the interval (a, b) the function f is concave down.

(c) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimum for the function $f(x)$.

(d) L'Hopital's rule applies to limit indeterminate forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

2. (16 pts) Compute each of the following limits (8 pts each):

(a) $\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}} \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} =$

$= \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to+\infty} \frac{2\sqrt{x}}{x\sqrt{x}}$

$= \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = \boxed{0}$

(b) $\lim_{x\to 0}(1+3x)^{1/x}$ 1^∞ (apply $e^{\ln}$ trick)

$= \lim_{x\to 0} e^{\ln\left((1+3x)^{\frac{1}{x}}\right)} =$

$= \lim_{x\to 0} e^{\frac{1}{x}\ln(1+3x)} =$ continuity of exponential function

$= e^{\lim_{x\to 0}\frac{\ln(1+3x)}{x}} = \boxed{e^3}$

$\lim_{x\to 0} \frac{\ln(1+3x)}{x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to 0} \frac{\frac{1}{1+3x}\cdot 3}{1} = 3$

3. (24 pts) Find the indicated antiderivatives (6 pts each):

(a) $\int \left(3\sqrt{x} - \frac{e^x}{2} + \frac{1}{1+x^2}\right) dx$ $= \int \left(3x^{\frac{1}{2}} - \frac{1}{2}e^x + \frac{1}{1+x^2}\right) dx$

$= \cancel{3} \cdot \frac{2}{\cancel{3}} x^{\frac{3}{2}} - \frac{1}{2} e^x + \arctan x + c$

(b) $\int \frac{x^2}{2x^3+1}\, dx$ $=$

sub. $u = 2x^3 + 1$

$du = 6x^2\, dx$

$\frac{1}{6} du = x^2\, dx$

$\int \frac{\frac{1}{6} du}{u} = \frac{1}{6} \int \frac{1}{u}\, du$

$= \frac{1}{6} \ln|u| + c$

$\boxed{= \frac{1}{6} \ln|2x^3+1| + c}$

(c) $\int \sin^4(5t)\cos(5t)\, dt$ $=$

sub. $u = \sin(5t)$

$du = \cos(5t) \cdot 5\, dt$

$\frac{1}{5} du = \cos(5t)\, dt$

$\int u^4 \cdot \frac{1}{5} du = \frac{1}{5} \cdot \frac{1}{5} u^5 + c$

$\boxed{= \frac{1}{25} \sin^5(5t) + c}$

(d) $\int x\sqrt{4-x}\, dx$ $=$

sub $u = 4 - x \Rightarrow x = 4 - u$

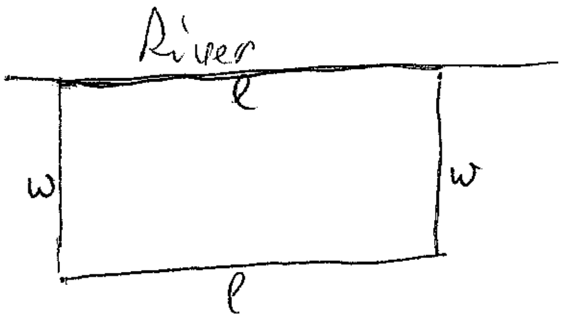
$dx = -du$

$= \int (4-u)\sqrt{u} \cdot (-du) = -\int \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$

$= -\left(4 \cdot \frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}}\right) + c$

$\boxed{= -\frac{8}{3}(4-x)^{\frac{3}{2}} + \frac{2}{5}(4-x)^{\frac{5}{2}} + c}$

4. (14 pts) Suppose you are allowed to choose a rectangular plot of land along a (straight) river. The rectangular plot is to have an area of 3000 square meters. You are required to fence in your land using two kinds of fencing. Three of the four sides will use heavy-duty fencing selling for \$30 per meter while the remaining side (along the river) will use standard fencing selling for \$10 per meter. How should you choose the dimensions of your plot of land in order to minimize the cost of fencing? (It's OK if your result contains square-roots.)



Let l and w be the length and width of the plot

Assume l is along the river

(but it's fine if you assume w along the river)

$$\text{Area} = l \cdot w = 3000$$

$$C = 10 \cdot l + 30 \cdot (l + 2w) = 10l + 30l + 60w = 40l + 60w$$

↑ cost

From $l \cdot w = 3000$ we solve for one variable (say l)

$l = \frac{3000}{w}$ and substitute in the cost formula

$$C(w) = 40 \cdot \frac{3000}{w} + 60w = 120{,}000\, w^{-1} + 60w$$

we want to find the absolute minimum for $C(w)$ when $w \in (0, +\infty)$

$$C'(w) = -120{,}000\, w^{-2} + 60 = 60 - \frac{120{,}000}{w^2}$$

$$C'(w) = 0 \iff 60 = \frac{120{,}000}{w^2} \iff w^2 = 2000$$

so $w = \sqrt{2000} = 10\sqrt{20} = \underline{20\sqrt{5}}$

This critical pt. is an abs. minimum as $C''(w) = 240{,}000\, w^{-3} > 0$ when $w \in (0, +\infty)$

Thus, the dimensions for minimum cost are

$\boxed{w = 20\sqrt{5}}$ and $\boxed{l = \frac{300\cancel{0}}{2\cancel{0}\sqrt{5}} = \frac{150}{\sqrt{5}} = \frac{150\sqrt{5}}{5} = 30\sqrt{5}}$

5. (12 pts) Find all critical points of the function $f(x) = x^4 - \frac{2}{3}x^3$ and determine their type (local minimum, local maximum, or neither).

$$f'(x) = 4x^3 - \frac{2}{3}\cdot 3x^2 = 4x^3 - 2x^2 = 2x^2(2x-1)$$

$$f'(x) = 0 \iff \boxed{x=0} \text{ or } \boxed{x=\frac{1}{2}} \leftarrow \text{critical points}$$

To determine their type, do either a sign chart for $f'(x)$ or apply the second derivative test. First way is better as the second derivative test is inconclusive for $x=0$

x	$-\infty$ -1 0 $\frac{1}{2}$ $+\infty$
$f'(x)$	- - - - 0 - - - 0 + + + + +
$f(x)$	$\searrow$ $\searrow$ $\nearrow$

Thus $x=\frac{1}{2}$ is a local minimum while $x=0$ is neither a local min nor a local max.

In fact, $x=0$ is a critical and inflection point for $f(x)$

6. (14 pts) A baseball is thrown straight upward from ground level with an initial velocity of 96 ft/s.

(a) (8 pts) Use integration to find the formulas for the velocity $v(t)$ and the position $s(t)$ of the baseball at time t. Assume gravitational acceleration $g = -32$ ft/s^2.

(b) (6 pts) When does the ball reach the maximum height?

(a) $a = -32$ $\quad v(t) = \int a\,dt = \int -32\,dt = -32t + v_0$

$$\boxed{v(t) = -32t + 96}$$

$$s(t) = \int v(t)\,dt = \int(-32t + 96)\,dt$$

$$s(t) = -\frac{32t^2}{2} + 96t + s_0$$

As $s_0 = 0$, Thus $\boxed{s(t) = -16t^2 + 96t}$

(b) The ball reaches maximum height when $s'(t) = v(t) = 0$

so $-32t + 96 = 0$ so $t = \frac{96}{32} = 3$ s is the moment the ball has maximum height.

7. (18 pts) Sketch the complete graph of the function $f(x) = \frac{x}{(x+2)^2}$. Make sure to clearly indicate the domain, critical points and their nature, inflection points and asymptotes (justified with limits). Your work should also include a sign chart indicating the intervals where the function is increasing, decreasing, concave up, concave down. To ease your task, here are the formulas for the first and second derivative:

$$f'(x) = \frac{2-x}{(x+2)^3}, \quad f''(x) = \frac{2(x-4)}{(x+2)^4}.$$

Domain : all reals, except $x=-2$

Suspect a vertical asymptote at $x=-2$

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x}{(x+2)^2} = \frac{-2}{0^+} = \underline{\underline{-\infty}}$, also $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x}{(x+2)^2} = \underline{\underline{-\infty}}$

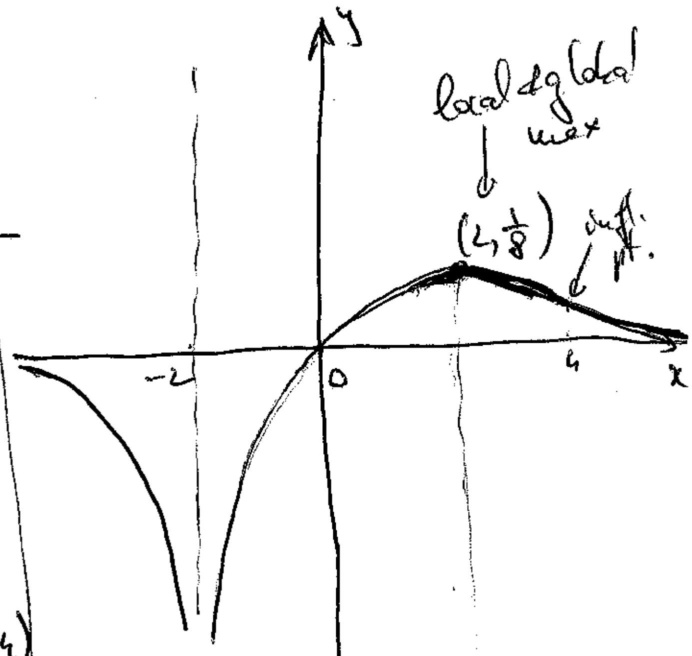
so $\underline{x=-2 \text{ is a vertical asymptote}}$

End behavior / Horiz. asymptotes

$\lim_{x\to \pm\infty} f(x) = \lim_{x\to \pm\infty} \frac{x}{(x+2)^2} \underset{\text{Lead Term}}{=} \lim_{x\to \pm\infty} \frac{x}{x^2} = 0$ so $\underline{y=0}$ is a H.A. both when $x\to+\infty$ and $x\to-\infty$.

Sign chart

x	$-\infty$		-2		2		4		$+\infty$
$f'(x)$		$- - - -$	‖	$+ + +$	0	$- - -$		$- - -$	
$f(x)$	0	$\searrow$	$-\infty$ ‖ $-\infty$	$\nearrow$	$f(2)=\frac{2}{4^2}=\frac{1}{8}$	$\searrow$		$\searrow$	0
$f''(x)$		$- - - -$	‖	$- - -$		$- -$	0	$+ + + +$	



$f(x)$ is decreasing on the intervals $(-\infty, -2)$ and $(2, +\infty)$

$f(x)$ is increasing on $(-2, 2)$

$f(x)$ is concave down on $(-\infty, -2)$ and on $(-2, 4)$

$f(x)$ is concave up on $(4, +\infty)$

$x=2$ is a local and global max. ; $x=4$ is an inflection point