Worksheet 01/22 Calculus I Spring 2019 **Group nr:** ______

1. Consider the function $f(x) = \frac{8-2x^2}{x^2-6x+8} = \frac{-2(x^2-4)}{(x-4)(x-2)}$

(a) Determine the points of discontinuity for $f(x)$.

Find vertical asymptotes:

$x^2-6x+8=0 \Rightarrow (x-4)(x-2)=0 \Rightarrow$ $\boxed{x=4 \text{ and } x=2 \text{ are points of discontinuity}}$

(b) Use limits to understand the behavior of the function near the points of discontinuity. Are any of these removable discontinuities?

Note: If the limit exists and is not $\pm\infty$, where the function is discontinuous, it is a removable discontinuity.

$\lim_{x\to 2^-} \frac{-2(x^2-4)}{(x-4)(x-2)} \stackrel{?}{=} \lim_{x\to 2^+} \frac{-2(x^2-4)}{(x-4)(x-2)}$

$\lim_{x\to 2^-} \frac{-2\cancel{(x-2)}(x+2)}{(x-4)\cancel{(x-2)}} \stackrel{?}{=} \lim_{x\to 2^+} \frac{-2\cancel{(x-2)}(x+2)}{(x-4)\cancel{(x-2)}}$

$\lim_{x\to 2^-} \frac{-2(x+2)}{x-4} \stackrel{?}{=} \lim_{x\to 2^+} \frac{-2(x+2)}{x-4}$;

$\frac{-\cancel{2}(4)}{-\cancel{2}} \stackrel{?}{=} \frac{-\cancel{2}(4)}{-\cancel{2}}$

$4 \stackrel{\checkmark}{=} 4$, $\boxed{\text{removable discontinuity @}x=2}$

$\lim_{x\to 4^-} \frac{-2(x+2)}{x-4} \stackrel{?}{=} \lim_{x\to 4^+} \frac{-2(x+2)}{x-4}$

$\frac{-2(6)}{-0} \stackrel{?}{=} \frac{-2(6)}{+0}$

$\infty \neq -\infty$, DNE, $\boxed{\text{nonremovable discontinuity @ } x=4}$

(c) Does this function have vertical asymptotes? Briefly justify your answer.

Yes, at $\boxed{x=4}$ where the limits on both sides approach $\pm\infty$. In this case, the limit does not exist.

(d) Does this function have horizontal asymptotes? Justify your answer with limits.

Yes, this function has horizontal asymptote(s),

$\lim_{x\to\infty} \frac{8-2x^2}{x^2-6x+8} \stackrel{\text{Jonk Rule}}{=} \frac{-2\cancel{x^2}}{\cancel{x^2}} = -2$; $\lim_{x\to-\infty} \frac{8-2x^2}{x^2-6x+8} \stackrel{\text{Jonk Rule}}{=} \frac{-2\cancel{x^2}}{\cancel{x^2}} = -2 \Rightarrow$ $\boxed{\text{The function has a horizontal asymptote at } y=-2.}$

2. Find, if possible, a value for the constant $k \geq 0$ which will make the function $g(x)$ continuous at $x = 0$.

$$g(x) = \begin{cases} \frac{1-\cos(kx)}{x^2} & \text{if } x < 0 \\ 1+\sin(3x) & \text{if } x \geq 0 \end{cases},$$

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x)$

$\Downarrow$

$\lim_{x\to 0^-} \frac{1-\cos(kx)}{x^2} = \lim_{x\to 0^+} 1+\sin(3x)$

$\lim_{x\to 0^-} \frac{(1-\cos kx)}{x^2} \frac{(1+\cos kx)}{(1+\cos kx)} = 1$

$\lim_{x\to 0^-} \frac{1-\cos^2 kx}{x^2(1+\cos kx)} = 1$

$\lim_{x\to 0^-} \frac{\sin^2 kx}{x^2} \cdot \frac{1}{1+\cos kx} = 1$

$\lim_{x\to 0^-} \left(\frac{\sin kx}{x}\right)^2 \cdot \frac{1}{1+\cos kx} = 1$

$k^2 \cdot \frac{1}{1+\cos 0} = 1$

$k^2 \cdot \frac{1}{1+1} = 1$

$\frac{k^2}{2} = 1 \Rightarrow k^2 = 2$

$\boxed{k = \sqrt{2}}$

3. True or False questions. Answer and briefly justify your answer in each case.

(i) If $f(x)$ is a continuous function and $\lim_{x\to 3} f(x) = 4$ then $f(3) = 4$ **True** **False**

Justification:

Since $f(x)$ is continuous, $\lim_{x\to a} f(x) = f(a)$ for all a

(ii) $\lim_{x\to+\infty} \cos\left(\frac{\pi x^2}{2x^2+1}\right) = 0$ **True** **False**

Justification:

$\lim_{x\to\infty} \cos\left(\frac{\pi x^2}{2x^2+1}\right) = \cos\left(\lim_{x\to\infty} \frac{\pi x^2}{2x^2+1}\right) \overset{\text{Junk Rule}}{=} \cos\left(\frac{\pi x^2}{2x^2}\right) = \cos\frac{\pi}{2} = 0$

(iii) The function $f(x) = \frac{x}{\sqrt{x^2+1}}$ is defined and is continuous for all real numbers x. **True** **False**

Justification:

- $\sqrt{x}$ has a domain of $[0,\infty)$
- $\frac{1}{x}$ has a domain of $(-\infty,0)\cup(0,\infty)$

$\Rightarrow \frac{1}{\sqrt{x}}$ has a domain $(0,\infty) \Rightarrow \sqrt{x^2+1}$ can never be 0 or negative, no discontinuities occur

(iv) The function $f(x) = \sec x$ is defined and is continuous for all real numbers x. **True** **False**

Justification:

$\sec x = \frac{1}{\cos x} \Rightarrow$ $\cos x = 0$ $x = \frac{\pi}{2} \pm 2k\pi, \frac{3\pi}{2} \pm 2k\pi$; k is an integer ; Vertical asymptotes (nonremovable discontinuities) occur where $\cos x = 0$

(v) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ **True** **False**

Justification:

Let $f(x) = \sin x$ and $g(x) = x$ $\lim_{x\to 0} \frac{\sin x}{x} = 1$, not every limit with indeterminate forms equals 0.

4. The temperature in an oven is given by $T(t) = t^3 + 3t^2 + 5$, where t is given in minutes, and $T(t)$ is given in Celsius degrees. Find, to within 30 seconds, the moment when the temperature has reached one hundred degrees. (A calculator may be necessary.) What Calculus theorem are you using?

$t^3 + 3t^2 + 5 = 100$

Find an interval of width 0.5 minutes (30 seconds) that contains an x^* such that $f(x^*) = 100$.

Test intervals

$[0, 0.5]$

$T(0) = 5$ $T(0.5) = 5.875$

$[0.5, 1]$

$T(1) = 9$

$[1, 1.5]$

$T(1.5) = 15.125$

$[1.5, 2]$

$T(2) = 25$

$[2, 2.5]$

$T(2.5) = 39.375$

$[2.5, 3]$

$T(3) = 59$

$[3, 3.5]$

$T(3.5) = 84.625$

$[3.5, 4]$

$T(4) = 117$

Since $T(t)$ is continuous, by Intermediate Value Theorem (IVT), there exists a value x^* in the interval $[3.5, 4]$ such that $f(x^*) = 100$.