Solution Key

NAME: ______________________ Note: y' is the same as $\frac{dy}{dx}$.

MAC 2311: Worksheet #13

1) For each of the following implicitly defined functions, find $\frac{dy}{dx}$:

a) $y^4 - 3y^3 - x = 3$ at $(x, y) = (-5, 1)$.

$$\frac{d}{dx}(y^4-3y^3-x) = \frac{d}{dx}(3)$$
$$4y^3y' - 9y^2y' - 1 = 0$$
$$4y^3y' - 9y^2y' = 1$$
$$\Rightarrow y'(4y^3-9y^2) = 1$$
$$y' = \frac{1}{4y^3-9y^2} \Rightarrow \frac{dy}{dx} = \frac{1}{4y^3-9y^2}$$
$$\left.\frac{dy}{dx}\right|_{(-5,1)} = \frac{1}{4-9} = \boxed{-\frac{1}{5}}$$

b) $\cos(xy) = x - y$

$$\frac{d}{dx}(\cos(xy)) = \frac{d}{dx}(x-y)$$
$$-\sin(xy)\cdot(xy)' = 1-y'$$
$$-\sin(xy)(y+xy') = 1-y'$$
$$-y\sin(xy) - xy'\sin(xy) = 1-y'$$
$$\Rightarrow y' - xy'\sin(xy) = 1 + y\sin(xy)$$
$$y'(1-x\sin(xy)) = 1+y\sin(xy)$$
$$\boxed{\frac{dy}{dx} = \frac{1+y\sin(xy)}{1-x\sin(xy)}}$$

2) Consider the function implicitly defined by $y^4 = x + y$.

a) Find an expression for the derivative $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^4) = \frac{d}{dx}(x+y)$$
$$4y^3\cdot y' = 1 + y'$$
$$4y^3y' - y' = 1$$
$$\Rightarrow y'(4y^3-1) = 1$$
$$\boxed{\frac{dy}{dx} = \frac{1}{4y^3-1}}$$

b) Find the equation of the line tangent to this function at the point (0,1).

Point: $(0,1)$

Slope $= \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{3}$

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{4-1} = \frac{1}{3}$$

Tangent Line

$$y - 1 = \frac{1}{3}(x-0)$$
$$\Downarrow$$
$$y = \frac{1}{3}x + 1$$

c) Find where the tangent line is vertical.

The tangent line is vertical when $\frac{dy}{dx}$ is undefined. For rational functions, this occurs when the numerator is not zero and the denominator is equal to 0.; $y^4 - y = x$

$$\frac{1}{4y^3-1} = \text{und.} \Rightarrow 4y^3 - 1 = 0$$
$$y^3 = \frac{1}{4}$$
$$y = \sqrt[3]{\tfrac{1}{4}}$$
$$x = \left(\tfrac{1}{4}\right)^{\frac{4}{3}} - \left(\tfrac{1}{4}\right)^{\frac{1}{3}} \Rightarrow$$

The tangent line is vertical at: $\left(\left(\tfrac{1}{4}\right)^{\frac{4}{3}} - \left(\tfrac{1}{4}\right)^{\frac{1}{3}}, \left(\tfrac{1}{4}\right)^{\frac{1}{3}}\right)$

3) Without using a calculator, compute the following:

a) $\log_2(8) = 3;\ 2^3 = 8$

b) $\log_5(\frac{1}{25}) = -2;\ 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

c) $\log_{1/3}(9) = -2;\ (\frac{1}{3})^{-2} = 3^2 = 9$

4) If $\log_b(A) = 5$, $\log_b(B) = 3$, and $\log_b(C) = 2$, compute

$$\log_b\left(\frac{A^2}{B^4C^3}\right) = \log_b(A^2) - [\log_b(B^4) + \log_b(C^2)]$$
$$= \log_b(A^2) - \log_b(B^4) - \log_b(C^2)$$
$$= 2\log_b A - 4\log_b B - 2\log_b C = 2(5) - 4(3) - 2(2)$$
$$= 10 - 12 - 4 = \boxed{-6}$$

5) Solve the equation $\log_2(x^2 + 1) = 1$.

$x^2 + 1 = 2$
$x^2 = 1$ $\Rightarrow$ $\boxed{x = \pm 1}$

6) Solve the equation $5^{3x} = 7$.

$\cancel{\log_5 5}^{3x} = \log_5 7$
$3x = \log_5 7$ $\Rightarrow$ $\boxed{x = \frac{1}{3}\log_5 7 \quad \text{or} \quad x = \frac{\ln 7}{3\ln 5}}$

7) Find each of the following derivatives. What are you using in each case?

(a) $\frac{d}{dx}(e^{7x}) = e^{7x} \cdot 7 = 7e^{7x}$

(b) $\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \cdot f'(x) = f'(x)e^{f(x)}$

(c) $\frac{d}{dx}(e^{\pi}) = e^{\pi} \cdot 0 = 0$

*We use chain rule in each case.

8) Use the trick that $2^x = e^{\ln(2^x)} = e^{x(\ln 2)}$, to find a formula for

$$\frac{d}{dx}(2^x)$$

$$\frac{d}{dx}(2^x) = \frac{d}{dx}(e^{x\ln 2}) = e^{x\ln 2} \cdot \ln 2 = \boxed{2^x \ln 2 = \frac{d}{dx}[2^x]}$$