Solution Key

Note:
$\cos^2\theta + \sin^2\theta = 1$
$\sin^2\theta = 1 - \cos^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$

Group nr. ____ NAMES: ________________

MAC 2281: Worksheet Feb. 26, 2019

1) Use method shown in class to obtain the formula for dy/dx when $y = \arccos x$.

$y = \arccos x$

$\cos y = \cancel{\cos(\arccos x)}$

$\cos y = x$

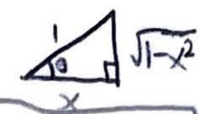
$(\cos y)' = (x)'$

$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = -\frac{1}{\sin y}$

$y = \arccos x$

$\frac{dy}{dx} = -\frac{1}{\sin(\arccos x)}$



$\boxed{\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}}$

2) Compute the derivative of each of the following functions. Simplify, if possible.

(a) $y = \arcsin(\cos(3x))$

$y' = \frac{1}{\sqrt{1-(\cos 3x)^2}} \cdot (\cos 3x)' \cdot (3x)'$

$y' = \frac{1}{\sqrt{\sin^2(3x)}} \cdot -3\sin 3x$

$y' = \frac{-3\sin \cancel{3x}}{\cancel{\sin 3x}} = \boxed{-3 = y'}$

(b) $y = x\arctan(\ln x)$

$y' = (x)'\arctan(\ln x) + x(\arctan(\ln x))'$

$y' = \arctan(\ln x) + x\left(\frac{1}{1+(\ln x)^2} \cdot (\ln x)'\right)$

$y' = \arctan(\ln x) + \cancel{x} \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{1}{\cancel{x}}$

$\boxed{y' = \arctan x + \frac{1}{1+(\ln x)^2}}$

3) Show that the function $f(x) = 2x^3 + 6x - 5$ is one to one and then find $(f^{-1})'(3)$. Note that $f(1) = 3$.

$f(x)$ is one-to-one if $f'(x)$ is strictly increasing or decreasing ($f'(x) > 0$ or $f'(x) < 0$).

$f'(x) = 6x^2 + 6 \Rightarrow 6x^2 + 6 > 0$ ✓ $f(x)$ is one-to-one.

$f^{-1}(f(x)) = x$

$(f^{-1}(f(x)))' = (x)'$

$(f^{-1}(f(x)))' \cdot f'(x) = 1$

$(f^{-1}(f(x)))' = \frac{1}{f'(x)}$

$(f^{-1}(3))' = (f^{-1}(f(1)))' = \frac{1}{f'(1)} = \frac{1}{6(1)^2+6} = \boxed{\frac{1}{12}}$

equivalent

4) (Lysis of a bacterium): A spherical bacterium has its cell wall perforated. As a result, water flows into the bacterium at 100 cubic micrometers per second. $\Leftrightarrow \frac{dV}{dt} = 100\ \mu m^3/sec$

a) What is the rate of change of the radius at the instant that the radius is $\underbrace{30 \text{ micrometers}}_{r = 30\ \mu m}$?

$$V_{sphere} = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$$

$$\left.\frac{dr}{dt}\right|_{(r=30)} = \frac{100}{4\pi(30)^2} = \frac{10\cancel{0}}{4\pi(90\cancel{0})}$$

$$\boxed{\left.\frac{dr}{dt}\right|_{(r=30)} = \frac{1}{36\pi}\ \mu m/sec}$$

b) What is the rate of change of the surface area at the instant that the radius is $\underbrace{30 \text{ microm-}}_{r = 30\ \mu m}$ eters?

$$S_{sphere} = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi(30)\left.\frac{dr}{dt}\right|_{(r=30)}$$

$$\frac{dS}{dt} = \frac{240\cancel{\pi}}{36\cancel{\pi}}$$

$$\frac{dS}{dt} = \frac{40}{6}$$

$$\boxed{\frac{dS}{dt} = \frac{20}{3}\ \mu m^2/sec}$$

5) Suppose that a cylindrical tank is being filled with water at a rate of 100 cm³/min. $\left(\frac{dV}{dt} = 100\ cm^3/min\right)$ If the tank has a radius of 50 cm, at what rate is the height of water in the tank changing when the tank is 100 cm full?

$r = 50cm$, this is fixed (no need to treat r as a variable)

$h = 100cm$

$$V_{cylinder} = \pi r^2 h = \pi(50)^2 h$$

$$\frac{dV}{dt} = \pi(2500)\frac{dh}{dt}$$

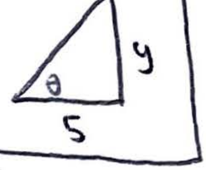
$$\frac{dh}{dt} = \frac{dV/dt}{2500\pi}$$

$$\left.\frac{dh}{dt}\right|_{(h=100)} = \frac{1\cancel{00}}{25\cancel{00}\pi}$$

$$\boxed{\left.\frac{dh}{dt}\right|_{(h=100)} = \frac{1}{25\pi}\ cm/min}$$

6) A telescope on the ground is tracking a rocket which is rising vertically from a launchpad. The telescope is 5 kilometers from the launchpad and denote by θ the angle with respect to which the telescope observes the rocket above the ground. Suppose that at the moment when the rocket is 10km above the ground ($y = 10\ km$), the angle θ is increasing at a rate of one degree per second. What is the vertical speed of the rocket at that moment?

$\frac{d\theta}{dt} = 1°/sec \Downarrow \frac{d\theta}{dt} = \frac{\pi}{180}\ rad/sec$



$$\tan\theta = \frac{y}{5}$$

$$\cancel{\arctan}(\tan\theta) = \arctan\left(\frac{y}{5}\right)$$

$$\theta = \arctan\left(\frac{y}{5}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{y}{5}\right)^2}\cdot\left(\frac{y}{5}\right)'$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{y}{5}\right)^2}\cdot\left(\frac{dy}{dt}\cdot\frac{1}{5}\right)$$

$$\frac{d\theta}{dt} = \frac{dy/dt}{5+5\left(\frac{y^2}{25}\right)}$$

$$\frac{dy}{dt} = \frac{d\theta}{dt}\left(5+\frac{y^2}{5}\right)$$

$$\left.\frac{dy}{dt}\right|_{(y=10)} = \left(\frac{\pi}{180}\right)\left(5+\frac{10^2}{5}\right)$$

$$= \frac{\pi}{180}\left(5+\frac{\cancel{100}^{20}}{\cancel{5}}\right)$$

$$= \frac{25\pi}{180}$$

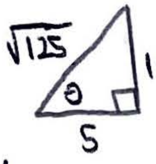
$$\boxed{\left.\frac{dy}{dt}\right|_{(y=10)} = \frac{5\pi}{36}\ km/sec}$$

OR

$$\tan\theta = \frac{y}{5}$$

$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{dy}{dt}$$

$$\frac{dy}{dt} = 5\sec^2\theta\,\frac{d\theta}{dt}$$



$$\Rightarrow \sec^2\theta = \left(\frac{\sqrt{125}}{5}\right)^2 = \frac{125}{25}$$

$$\sec^2\theta = 5$$

$$\left.\frac{dy}{dt}\right|_{(y=10)} = 5(5)\left(\frac{\pi}{180}\right) = \frac{25\pi}{180}$$

$$\boxed{\left.\frac{dy}{dt}\right|_{(y=10)} = \frac{5\pi}{36}\ km/sec}$$