Geometric series theorem:

Given a geometric series, $\sum_{k=0}^{\infty} cr^k$, if $|r| < 1$ the series converges to $\dfrac{c}{1-r}$.

If $|r| \geq 1$, the geometric series diverges.

Proof: We start from establishing the following identity:

$$1 - r^{n+1} = (1-r)(1 + r + r^2 + \ \dots \ + r^n) \ .$$

This is seen just by distributing the right hand-side and observing we get a telescopic pattern

$$(1-r)(1+r+r^2+ \dots +r^n) = (1-r)\cdot 1+(1-r)r+(1-r)r^2+ \dots +(1-r)r^n = 1-r+r-r^2+r^2-r^3+ \dots +r^n-r^{n+1} = 1-r^{n+1} \ .$$

By definition, the convergence or divergence of the series is determined by the convergence or divergence of its sequence of partial sums:

$$S_n = \sum_{k=0}^{n} cr^k = c + cr + \dots + cr^n \ .$$

If $r = 1$, $S_n = (n+1)c$, so for $c \neq 0$, the limit of S_n is infinite, so S_n and hence the series diverges.
Next we treat the case $r \neq 1$. In the formula for S_n, factoring c and using the above identity, we get

$$S_n = c\frac{1 - r^{n+1}}{1 - r} \ .$$

We know that if $|r| < 1$, then

$\lim\limits_{n \to +\infty} r^{n+1} = 0,$ and thus S_n converges and $\sum_{k=0}^{\infty} cr^k = \lim\limits_{n \to +\infty} S_n = c\dfrac{1}{1-r}$.

If $r > 1$, $\lim\limits_{n \to +\infty} r^{n+1} = +\infty,$ so S_n and the series are divergent to $sign(c)\infty$.

If $r \leq -1$, $\lim\limits_{n \to +\infty} r^{n+1}$ does not exist, so the limit of S_n does not exist.

Thus the series diverges if $r \leq -1$, or $r \geq 1$. All cases have been proved. QED