Name: Solution Key Panther ID: ____________

Exam 1 Calculus II Spring 2009

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (10 pts) Circle "True" or "False" for each part. You do not have to explain.

(a) If $f(x)$ is integrable on $[a,b]$ and $f(x) \leq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \leq 0$.

True False

As $f(x) \leq 0$ any Riemann sum and hence the integral must be ≤ 0

(b) $(1+2+3+\ldots+n)^2 = 1^2+2^2+3^2+\ldots+n^2$

True False

See the closed-form formulas in 5.4, or observe that the left hand side is strictly larger

(c) The function $f(x) = 1/x$ is integrable on $[0,1]$.

True False

$f(x)=\frac{1}{x}$ is not bounded on $[0,1]$

(d) If the acceleration of a particle is zero on the time interval $[t_1, t_2]$, then the displacement of the particle is zero on $[t_1, t_2]$.

True False

The particle could move with constant, non-zero velocity

(e) $\frac{d}{dx}\left(\int_1^{x^2} \frac{\sin t}{t}\,dt\right) = \frac{2\sin(x^2)}{x}$

True False

Combination of FTC (part 2) and chain rule.

2. (10 pts) Find the exact value of each of the following sums

(a) (5pts) 1+3+5+7+ ... +99 = $\sum_{k=1}^{50}(2k-1) = 2\cdot\sum_{k=1}^{50} k - \sum_{k=1}^{50} 1 = \cancel{2}\frac{50\cdot 51}{\cancel{2}} - 50 = 50^2 = 2500$

Another solution (using Gauss' idea)

Let $S = 1+3+5+\ldots+97+99$
$S = 99+97+95+\ldots+3+1$
add the two $\Rightarrow 2S = \underbrace{100+100+\ldots+100}_{50 \text{ terms}} \Rightarrow$

$\Rightarrow S = \frac{100\cdot 50}{2} = 50^2 = \boxed{2500}$

(b) (5pts) $\sum_{k=1}^{99}\left(\frac{1}{2k} - \frac{1}{2k+2}\right) =$

Observe it is a telescopic sum, hence write it in expanded form:

$= \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}\right) + \cancel{\ldots} + \left(\cancel{\frac{1}{196}} - \cancel{\frac{1}{198}}\right) + \left(\cancel{\frac{1}{198}} - \frac{1}{200}\right) =$

$= \frac{1}{2}^{100} - \frac{1}{200} = \boxed{\frac{99}{200}}$

3. (8 pts) Circle "True" or "False", but this time also **carefully explain** your answer:

If $f(x)$ is a continuous function on $[a, b]$ and $F'(x) = f(x)$, then the average value of f over the interval $[a, b]$ is equal to the average rate of change of F on $[a, b]$. → This is a restatement of FTC (part a)

True   False

Explanation:

(1) Avg. of f on $[a,b]$ $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

(2) Avg. rate of change of F on $[a,b]$ $= \dfrac{F(b)-F(a)}{b-a}$

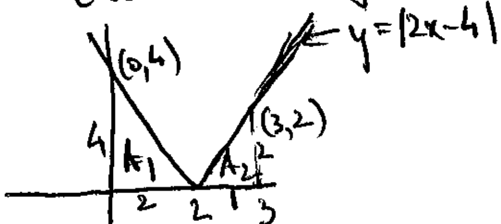
(3) FTC (part a) $\Rightarrow \displaystyle\int_a^b f(x)\,dx = F(b)-F(a)$

$\Rightarrow \dfrac{\int_a^b f(x)\,dx}{b-a} = \dfrac{F(b)-F(a)}{b-a}$

4. (30 pts) Compute each integral (6 pts each):

(a) $\displaystyle\int_0^3 |2x-4|\, dx$

Easiest solution: geometry



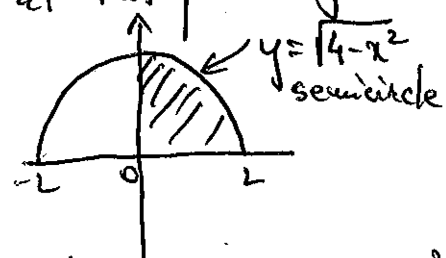
$\displaystyle\int_0^3 |2x-4|\,dx = A_1 + A_2 = \dfrac{4\cdot 2}{2} + \dfrac{1\cdot 2}{2} = \boxed{5}$

(b) $\displaystyle\int_0^{\pi/3} \sec x \tan x \, dx$ $= \sec x \Big|_{x=0}^{x=\frac{\pi}{3}} = \sec\frac{\pi}{3} - \sec 0 = \dfrac{1}{\cos\frac{\pi}{3}} - \dfrac{1}{\cos 0} = 2 - 1 = \boxed{1}$

(remember that $(\sec x)' = \sec x \tan x$)

(c) $\displaystyle\int_0^2 \sqrt{4-x^2}\, dx$

Easiest and only solution at this point: geometry



$\displaystyle\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi \cdot 2^2 = \boxed{\pi}$

(d) $\displaystyle\int_0^2 x\sqrt{4-x^2}\, dx$ $= *$

sub. $\boxed{w = 4-x^2}$

$dw = -2x\,dx \Rightarrow -\frac{1}{2}dw = x\,dx$

$* = \displaystyle\int_{w=4}^{w=0} \sqrt{w}\left(-\tfrac{1}{2}dw\right) = \frac{1}{2}\int_0^4 w^{\frac{1}{2}}\,dw$

$= \dfrac{1}{\cancel{2}}\cdot\dfrac{\cancel{2}}{3} w^{\frac{3}{2}}\Big|_0^4 = \dfrac{1}{3}\cdot 8 = \boxed{\dfrac{8}{3}}$

(e) $\displaystyle\int_0^{\pi/4} \frac{\sin(2x)}{2+\cos(2x)}\, dx$ $= *$

Sub. $w = 2+\cos(2x)$

$dw = -2\sin(2x)\,dx$

$\Rightarrow -\frac{1}{2}dw = \sin(2x)\,dx$

$* = \displaystyle\int_{w=3}^{w=2} \frac{-\frac{1}{2}dw}{w} = \frac{1}{2}\int_2^3 \frac{1}{w}\,dw$

$= \dfrac{1}{2}\ln|w|\Big|_{w=2}^{w=3} = \boxed{\dfrac{1}{2}(\ln 3 - \ln 2)}$

$= \boxed{\dfrac{1}{2}\ln\left(\dfrac{3}{2}\right)}$

5. (10 pts) A rock is dropped (with zero initial velocity) from the top of Green Library, at a height of 128 feet.

(a) (5 pts) Use integration to find a formula for its height in feet, $s(t)$, where t is the time in seconds since the rock was released.

(b) (5 pts) How long does it take for the rock to hit ground ? Assume gravitational acceleration -32 ft/sec^2.

(a) Let $a=-g$ be the constant gravitational acceleration and let $v(t)$ be the velocity at time t.

As $a = v'(t) \Rightarrow v(t) = \int a\,dt = \int -g\,dt = -gt + c$

But $c = v(0)$, so $\boxed{v(t) = -gt + v_0}$

Since $v(t) = s'(t) \Rightarrow s(t) = \int(-gt+v_0)dt = -\frac{gt^2}{2} + v_0 t + \tilde{c}$.

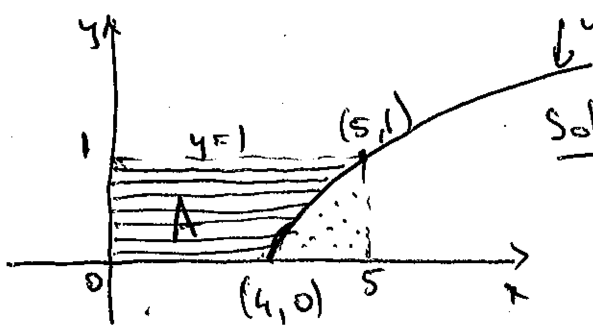
But $\tilde{c} = s(0)$, so $\boxed{s(t) = -\frac{gt^2}{2} + v_0 t + s_0}$

In our case, $v_0 = 0$, $s_0 = 128$, so $s(t) = -\frac{32t^2}{2} + 128 = -16t^2 + 128$

(b) $t = ?$ so that $s(t) = 0$

$0 = -16t^2 + 128 \Rightarrow t = \sqrt{8} = 2\sqrt{2}$ seconds

6. (10 pts) Find the area of the region bounded by $y = \sqrt{x-4}, y = 0, y = 1, x = 0$. Sketch of the region and computation are required.



The region is the one shaded "$\equiv$"

Sol.1: An easy way to find this area, is to subtract from the area of the rectangle $[0,5]\times[0,1]$ the area under the curve $y=\sqrt{x-4}$ on the interval $[4,5]$ (shaded with dots)

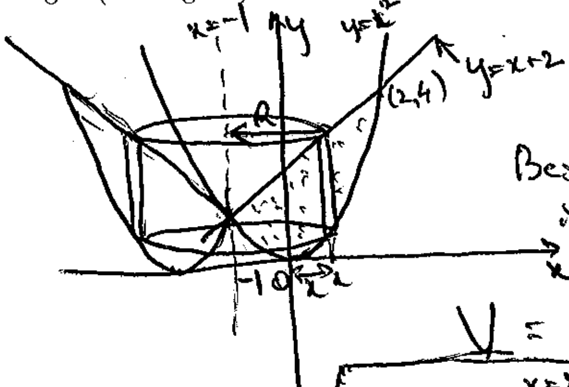
$$A = 5\times 1 - \int_4^5 \sqrt{x-4}\,dx = 5 - \int_4^5 (x-4)^{\frac{1}{2}}dx$$

$$A = 5 - \frac{2}{3}(x-4)^{\frac{3}{2}}\Big|_4^5 = 5 - \frac{2}{3} = \frac{13}{3}$$

Sol.2. Consider horizontal stripes and integrate w.r.t. y. since $y=\sqrt{x-4} \Rightarrow x = 4+y^2$

$$A = \int_{y=0}^{y=1} (4+y^2)dy = \left(4y + \frac{y^3}{3}\right)\Big|_{y=0}^{y=1} = 4 + \frac{1}{3} = \frac{13}{3}$$

7. (10 pts) The region bounded between $y = x^2$ and $y = x + 2$ is rotated around the line $x = -1$. Set up an integral (or integrals) for the volume of the solid (computation is **not** required, but a picture is).



Int. pts: $\begin{cases} y = x^2 \\ y = x+2 \end{cases} \Rightarrow x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x+1)(x-2) = 0$

$\Rightarrow \boxed{x=-1}, \boxed{x=2}$

Best method for volume in this case: <u>Cylindrical shells</u>

$$V = \int_?^? 2\pi R \cdot h \cdot Th$$

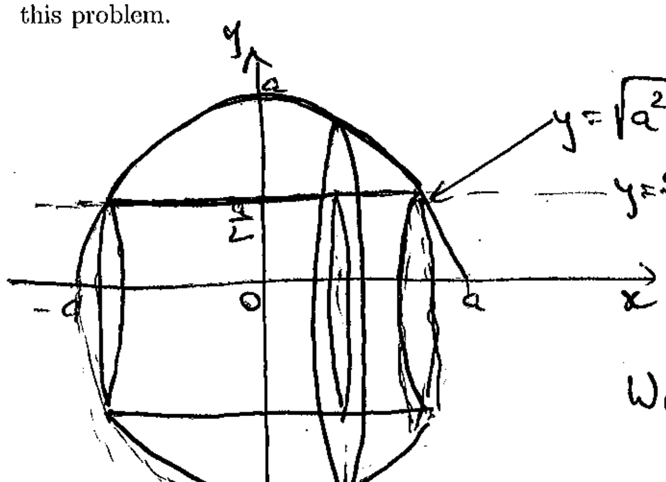
$Th = dx$

$R = 1 + x$

$h = x + 2 - x^2$

$$\boxed{V = 2\pi \int_{x=-1}^{x=2} (1+x)(x+2-x^2)\,dx}$$

8. (12 pts) The region bounded between $y = \sqrt{a^2 - x^2}$ and $y = \frac{a}{2}$, where a is a positive constant, is rotated around the x-axis. Find the volume of the solid obtained. Sketch of the solid and computation are required for this problem.



Easiest method is, in this case, the slicing method, but the cyl. shells method could be used as well

With the slicing method

$$V = \int_?^? A_{slice} \cdot Th$$

$Th = dx$

$A_{slice} = \pi R^2 - \pi r^2 = \pi (R^2 - r^2)$

$R = \sqrt{a^2 - x^2} \qquad r = \frac{a}{2}$

Limits of integration ← intersect $y = \sqrt{a^2 - x^2}$ with $y = \frac{a}{2} \Rightarrow$

$\Rightarrow \frac{a}{2} = \sqrt{a^2 - x^2} \Rightarrow \frac{a^2}{4} = a^2 - x^2 \Rightarrow x^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4} \Rightarrow x = \pm \frac{\sqrt{3}a}{2}$

$$V = \int_{-\frac{\sqrt{3}a}{2}}^{\frac{\sqrt{3}a}{2}} \pi \left[\left(\sqrt{a^2 - x^2}\right)^2 - \left(\frac{a}{2}\right)^2\right] dx \underset{\text{symmetry}}{=} 2\pi \int_0^{\frac{\sqrt{3}a}{2}} \left[a^2 - x^2 - \frac{a^2}{4}\right] dx = 2\pi \int_0^{\frac{\sqrt{3}a}{2}} \left(\frac{3a^2}{4} - x^2\right) dx = *$$

$$* = 2\pi\left(\frac{3a^2}{4}x - \frac{x^3}{3}\right)\Big|_0^{\frac{\sqrt{3}a}{2}} = 2\pi \frac{\sqrt{3}a}{2}\left(\frac{3a^2}{4} - \frac{a^2}{4}\right) = \boxed{\pi\sqrt{3}a^3 \ldots}$$