Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus II Spring 2013

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) Circle the correct answer (work is not required for this problem) :

(a) The arc length of $y = \sin x$ from $x = 0$ to $x = \pi$ is given by

(i) $\dfrac{\sin \pi - \sin 0}{\pi - 0}$ (ii) $\displaystyle\int_0^{\pi} \sin x\, dx$ (iii) π (iv) $\displaystyle\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx$ (v) $\displaystyle\int_0^{\pi} \sqrt{1+\sin^2 x}\, dx$

(b) The expression

$$\frac{d}{dx}\left(\int_1^{x^2} \frac{\sin t}{t}\, dt\right) \text{ is equivalent to}$$

(i) $\dfrac{\sin(x^2)}{x^2}$ (ii) $\sin 1$ (iii) $\dfrac{2\sin(x^2)}{x}$ (iv) $2x\dfrac{\sin t}{t}$ (v) 0

(c) The average value of the function $f(x)$ over the interval $[a, b]$ is

(i) $f(\frac{a+b}{2})$ (ii) $\dfrac{f(a)+f(b)}{2}$ (iii) $f'(b) - f'(a)$ (iv) $\dfrac{f(b)-f(a)}{b-a}$ (v) 0

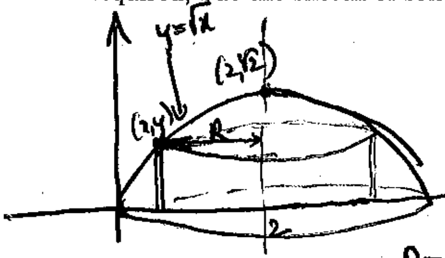
None is correct; (iv) works if "f" is replaced by "F", where F is an anti-derivative of "f".

(d) The partial fraction decomposition for $\frac{x+3}{x^4+9x^2}$ is of the form:

(i) $\dfrac{A}{x^2} + \dfrac{B}{x^2+9}$ (ii) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3} + \dfrac{D}{(x+3)^2}$ (iii) $\dfrac{x+3}{x^4} + \dfrac{x+3}{9x^2}$

(iv) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+9}$ (v) $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3}$

2. (12 pts) The region bounded by the curve $y = \sqrt{x}$, the x-axis and the vertical line $x = 2$ is revolved around the line $x = 2$. Set up an integral to express the volume of the solid obtained. (Computation of the integral is NOT required, but the sketch of solid is.)



Sol. 1 - with slicing

$V = \int_0^{?} A_{slice} \cdot Th$

$Th = dy \quad A_{slice} = \pi R^2$

$R = 2 - x = 2 - y^2$

$\boxed{V = \int_0^{\sqrt{2}} \pi (2-y^2)^2\, dy}$

Sol. 2 - with cyl. shells

$V = \int_0^{?} 2\pi R \cdot h \cdot Th$

$Th = dx \quad R = 2 - x \quad h = y = \sqrt{x}$

$\boxed{V = \int_0^2 2\pi (2-x)\sqrt{x}\, dx}$

3. (40 pts) Evaluate (10 pts each)

(a) $\int x^3 \ln(x)\, dx$ integration by parts

$du = x^3 dx \qquad v = \ln x$

$u = \frac{1}{4}x^4 \qquad dv = \frac{1}{x}dx$

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x} dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

$\boxed{= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$

(b) $\int \frac{1}{x^2 - 3x}\, dx$ partial fractions

$\frac{1}{x^2-3x} = \frac{1}{x(x-3)} = \frac{1}{3}\left(\frac{1}{x-3} - \frac{1}{x}\right)$

guess & check method

(you could do it with A, B et

$\int \frac{1}{x^2-3x} dx = \frac{1}{3}\int \left(\frac{1}{x-3} - \frac{1}{x}\right) dx$

$\boxed{= \frac{1}{3}\left(\ln|x-3| - \ln|x|\right) + C}$

(c) $\int_1^\infty x e^{-x^2}\, dx$

$\int x e^{-x^2} dx = -\frac{1}{2}\int e^w dw = -\frac{1}{2}e^w + C$

$w = -x^2 \qquad\qquad \shortparallel$

$dw = -2x\,dx \qquad -\frac{1}{2}e^{-x^2} + C$

$\int_1^\infty x e^{-x^2} dx = \lim_{b\to\infty} \int_1^b x e^{-x^2} dx = \lim_{b\to\infty} \left(-\frac{1}{2}e^{-x^2}\right)\Big|_1^b$

$= \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}\right) = 0 + \frac{1}{2}e^{-1}$

$\shortparallel$

$\boxed{\frac{1}{2e}}$

(d) $\int \frac{1}{x^2\sqrt{25 - x^2}}\, dx$ trig. substitution

$x = 5\sin\theta \Rightarrow \sqrt{25-x^2} = 5\cos\theta$

$dx = 5\cos\theta\, d\theta$

$= \int \frac{\cancel{5\cos\theta}\, d\theta}{25\sin^2\theta\, \cancel{5\cos\theta}} = \frac{1}{25}\int \csc^2\theta\, d\theta$

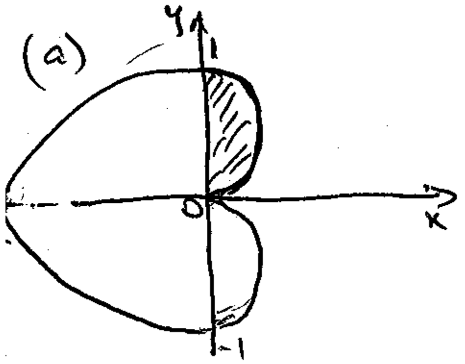
$= -\frac{1}{25}\cot\theta = -\frac{1}{25}\frac{\cancel{5}\cos\theta}{\cancel{5}\sin\theta} = -\frac{1}{25}\frac{5\cos\theta}{5\sin\theta}$

$\boxed{= -\frac{1}{25}\cdot\frac{\sqrt{25-x^2}}{x}}$

6. (14 pts) (a) (6 pts) Sketch the graph of the cardioid curve $r = 1 - \cos\theta$.

(b) (8 pts) Find the area of the region in the first quadrant that is inside the cardioid $r = 1 - \cos\theta$.

(a)



(b) $A = \int_{?}^{?} \frac{1}{2} r^2 \, d\theta \qquad A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 \, d\theta$

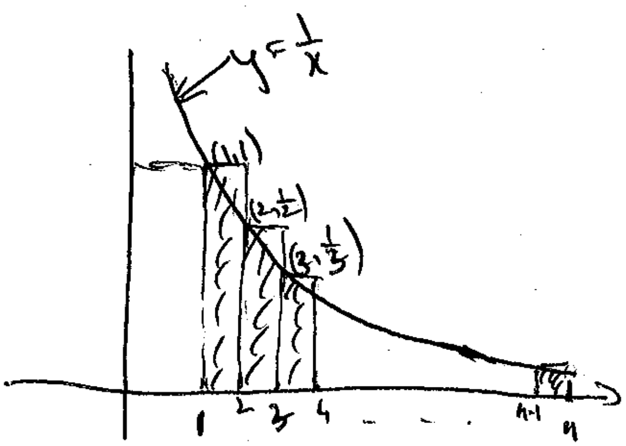
$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$

$= \dots = \boxed{\frac{3\pi}{8} - 1}$

7. (20 pts) (a) (7 pts) Let n be a positive integer. Find, in terms of n, the area below the graph of $f(x) = 1/x$ and above the x-axis on the interval $[1, n]$.

(b) (7 pts) Write the left-hand point Riemann sum approximation of the area in part (a) corresponding to the division of the interval $[1, n]$ into sub-intervals of length 1 (your answer should use the summation notation). Is this approximation an over or an under estimate of the area in part (a)? (A picture is required.)

(c) (6 pts) Use parts (a) and (b) to show that the harmonic series is divergent.



(a) $A = \int_1^n \frac{1}{x}\, dx = \ln x \Big|_{x=1}^{n} = \boxed{\ln n}$

(b) The L.R.S. is shaded in the picture

Its value is (note $\Delta x = 1$):

$LRS = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}$

This is an over-estimate of the area

(c) Thus $\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n-1} > \ln n$, so

$$\sum_{k=1}^{n-1} \frac{1}{k} > \ln n.$$

Taking the limit when $n \to +\infty$, as the left side is just the sequence of partial sums for the harmonic series, we get

$\sum_{k=1}^{\infty} \frac{1}{k} \geq \lim_{n\to\infty} \ln n = +\infty$. Thus $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$

8. (16 pts) Find the interval of convergence of the series

$\sum_{k=1}^{\infty} \frac{(-2)^k x^{3k}}{k^{3/2}}$. Determine whether any of the endpoints belong to the interval of convergence.

Apply Ratio Test to $\sum_{k=1}^{\infty} \left| \frac{(-2)^k x^{3k}}{k^{\frac{3}{2}}} \right| = \sum_{k=1}^{\infty} \frac{2^k |x|^{3k}}{k^{\frac{3}{2}}}$

$\frac{|x|^{3k+3}}{|x|^{3k}} = |x|^3$

$$\rho = \lim_{k \to +\infty} \frac{a_{k+1}}{a_k} = \lim_{k \to +\infty} \left(\frac{\cancel{2^{k+1}}^{\,2} \cdot |x|^{3(k+1)}}{(k+1)^{\frac{3}{2}}} \cdot \frac{k^{\frac{3}{2}}}{\cancel{2^k} \cdot |x|^{3k}} \right) =$$

$$\rho = \lim_{k \to +\infty} \left(2|x|^3 \cdot \frac{k^{\frac{3}{2}}}{(k+1)^{\frac{3}{2}}} \right) = 2|x|^3$$

So $\rho = 2|x|^3 < 1 \iff |x| < \frac{1}{\sqrt[3]{2}}$

End points: $x = \frac{1}{\sqrt[3]{2}}$ $\sum_{k=1}^{\infty} \frac{(-2)^k \left(\frac{1}{\sqrt[3]{2}}\right)^{3k}}{k^{\frac{3}{2}}} = \sum_{k=1}^{\infty} \frac{(-2)^k \cdot \frac{1}{2^k}}{k^{\frac{3}{2}}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\frac{3}{2}}}$ converges by AST

$x = -\frac{1}{\sqrt[3]{2}}$ ... $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges, p-series, $p = \frac{3}{2} > 1$. So $x \in \left[-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} \right]$

9. (14 pts) Use the MacLaurin series for e^x to find the MacLaurin series for $F(x) = \int e^{-x^2}\, dx$. Assume $F(0) = 0$.

MacLaurin series of $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Substituting x by $-x^2$, we get

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}.$$

Now we integrate this

$$F(x) = \int \left(\sum \frac{(-1)^k x^{2k}}{k!} \right) dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k+1} + c$$

Set $c = 0$, by the condition $F(0) = 0$