Name: Solution Key Panther ID: ____________

Exam 1 Calculus II Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) The first five terms of a sequence are

$$a_1 = \frac{1}{3},\ a_2 = -\frac{2}{5},\ a_3 = \frac{3}{7},\ a_4 = -\frac{4}{9},\ a_5 = \frac{5}{11}, \dots$$

(a) (4 pts) Assuming that the sequence follows the indicated pattern, find the formula for the general term a_n.

$$a_n = (-1)^{n+1} \cdot \frac{n}{2n+1} \quad \text{or} \quad a_n = (-1)^{n-1} \cdot \frac{n}{2n+1}$$

(b) (4 pts) Is the sequence a_n bounded? Justify your answer.

Yes. $|a_n| = \frac{n}{2n+1} < \frac{1}{2}$.

Thus $-\frac{1}{2} < a_n < \frac{1}{2}$ for all $n \geq 1$.

(c) (4 pts) Is the sequence a_n convergent? Justify your answer.

No. The subsequence $\{a_{2k+1}\}_k$ converges to $\frac{1}{2}$ as $k \to +\infty$ while the subsequence $\{a_{2k}\}_k$ converges to $-\frac{1}{2}$ as $k \to +\infty$

Note that $\lim_{n \to +\infty} \frac{n}{2n+1} = \frac{1}{2}$ so the sequence $\{|a_n|\}_n$ converges to $\frac{1}{2}$, but the sequence $\{a_n\}_n$ is divergent.

2. (18 pts) Evaluate each of the following (or show it diverges):

(a) $\lim_{n\to+\infty} n^2e^{-n}$ $= \lim_{n\to+\infty} \frac{n^2}{e^n}$ $\qquad \frac{\infty}{\infty}$

Using l'Hopital twice $\lim_{x\to+\infty} \frac{x^2}{e^x} = \lim_{x\to+\infty} \frac{2x}{e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = 0$

Thus $\boxed{\lim_{n\to+\infty} n^2 e^{-n} = 0}$

(b) $\sum_{k=2}^{+\infty} \frac{2^k}{3^k}$ geometric series with $r = \frac{2}{3} < 1$

$$\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k = \left(\frac{2}{3}\right)^2 \sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^{k-2} = \left(\frac{2}{3}\right)^2 \sum_{\ell=0}^{\infty} \left(\frac{2}{3}\right)^{\ell} = \frac{4}{9} \cdot \frac{1}{1-\frac{2}{3}} = \boxed{\frac{4}{3}}$$

Geom series Thm.

(c) $\sum_{k=1}^{+\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right)$

observe it is a telescopic series.

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \ldots +$$

$$+ \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \ldots\right)$$

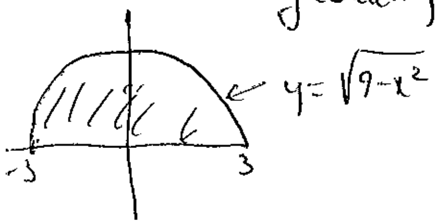
Thus $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$, so

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \boxed{\frac{3}{2}}$$

3. (30 pts) Compute each integral and simplify your answer when possible (6 pts each):

(a) $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{\pi \cdot 3^2}{2} = \frac{9\pi}{2}$

geometry



(b) $\int_0^1 \frac{1}{1+x^2}\,dx = \arctan x \Big|_{x=0}^{x=1}$

$= \arctan 1 - \arctan 0$

$= \boxed{\frac{\pi}{4}}$

(c) $\int_0^{\ln 3} \frac{e^x}{e^x+4}\,dx =$

subst. $u = e^x + 4$

$du = e^x \cdot dx$

$= \int_{u=5}^{u=7} \frac{du}{u} = \ln u \Big|_{u=5}^{u=7} =$

$= \boxed{\ln 7 - \ln 5}$

(d) $\int_0^2 x\sqrt{1+2x^2}\,dx =$

subst. $w = 1+2x^2$

$dw = 4x\,dx$

$\frac{1}{4}dw = x \cdot dx$

$= \int_{w=1}^{w=9} \sqrt{w} \cdot \frac{1}{4}\,dw = \frac{1}{4}\int_{w=1}^{w=9} w^{\frac{1}{2}}\,dw$

$= \frac{1}{4} \cdot \frac{2}{3} w^{\frac{3}{2}} \Big|_{w=1}^{w=9} = \frac{1}{6}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$

$= \frac{1}{6}\left(\left(\sqrt{9}\right)^3 - 1\right) = \frac{26}{6} = \boxed{\frac{13}{3}}$

(e) $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x\,dx =$

subst. $w = \sin x$

$dw = \cos x\,dx$

$= \int_{w=0}^{w=\frac{\sqrt{2}}{2}} w^3\,dw = \frac{1}{4} w^4 \Big|_{w=0}^{w=\frac{\sqrt{2}}{2}}$

$= \frac{1}{4} \cdot \left(\frac{\sqrt{2}}{2}\right)^4 = \frac{1}{4} \cdot \frac{1}{4} = \boxed{\frac{1}{16}}$

4. (20 pts) In each case answer True or False and give a brief justification (4 pts each)

(a) A convergent sequence must be monotone.

False : Example $a_n = (-1)^n \frac{1}{n}$ converges to 0, but is not monotone

(b) The sequence $a_n = \sqrt{n} - 1000$ is eventually positive.

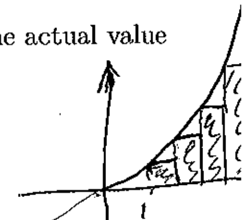
True ! For all $n > 10^6$, $\sqrt{n} - 1000 > 0$.

(c) The series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ converges to zero.

False : The sequence of partial sums $S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ so diverges

(d) Using a left end-point Riemann sum for $\int_1^5 x^3\, dx$ will produce an estimate which is less than the actual value of the integral.

True : Since $y = x^3$ is increasing (and positive) the left R. sum will be an underestimate of the integral



(e) If the function $f(x)$ is continuous on the interval $(1, 4)$ then $f(x)$ is integrable on $[1, 4]$.

False : If the function is continuous only on the open interval $(1,4)$, the function may not be bounded on $[1,4]$ and hence not integrable.

5. (10pts) (a) Find the average value of $f(x) = 1/x$ on the interval $[1, e]$.

$$f_{ave} = \frac{\int_1^e \frac{1}{x}dx}{e-1} = \frac{(\ln x)\Big|_{x=1}^{x=e}}{e-1} = \frac{\ln e - \ln 1}{e-1} = \frac{1}{e-1}$$

(b) Find a value of c so that $f(c) =$ the average value from part (a). Why such c is guaranteed to exist?

$f(x) = \frac{1}{x}$ is continuous on $[1,e]$, so by MVT for integrals there is a value $c \in [1,e]$ so that $f(c) = f_{ave}$.

The concrete value of c in this case:

$$\frac{1}{c} = \frac{1}{e-1} \Rightarrow \boxed{c = e-1}$$ Note that it is indeed in the interval $[1,e]$.

6. (10pts) Find the displacement and distance traveled by an object with velocity $v(t) = \cos(t)$, in feet per second, for $0 \le t \le \frac{3\pi}{2}$ seconds.

$$\text{Displacement} = \int_0^{\frac{3\pi}{2}} v(t)dt = \int_0^{\frac{3\pi}{2}} \cos(t)dt = \sin t \Big|_{t=0}^{t=\frac{3\pi}{2}} = \boxed{-1\ ft}$$

$$\text{Dist. traveled} = \int_0^{\frac{3\pi}{2}} |v(t)|dt = \int_0^{\frac{3\pi}{2}} |\cos t|dt = \int_0^{\frac{\pi}{2}} |\cos t|dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\cos t| d$$

$$= \int_0^{\frac{\pi}{2}} \cos t\, dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (-\cos t)dt = \sin t \Big|_0^{\frac{\pi}{2}} - (\sin t)\Big|_{t=\frac{\pi}{2}}^{t=\frac{3\pi}{2}} =$$

$$= (1-0) - (-1-1) = \boxed{3\ ft}$$

7. (12 pts) Choose ONE to prove. If possible, use sentences or formulas with complete justifications. The grading will be based on the clarity of your logic and explanations, as much as on any calculations involved.

(a) State and prove the geometric series theorem.

(b) State and prove the part of FTC about $\frac{d}{dx}(\int_a^x ...)$. You may use without proof MVT for integrals.

See your notes or textbook