Name: Solution Key Panther ID: ____________

FINAL EXAM Calculus II Fall 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (20 pts) In each case circle True or False. No justification needed. (2 pts each)

(a) Every continuous functions has an anti-derivative. **True** **False**

(b) Every bounded sequence is convergent. **True** **False**

(c) If f is positive and concave up on $[a, b]$, then $M_n \leq \int_a^b f(x)dx$. **True** **False**

(d) If f is strictly increasing on $[a, b]$, then $L_n < T_n < R_n$. **True** **False**

(e) The average value of f on $[a, b]$ is $\frac{f(a)+f(b)}{2}$. **True** **False**

(f) If $0 < a_k < \frac{1}{k}$ for all $k \geq 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

(g) If $0 < a_k < \frac{1}{k}$ for all $k \geq 1$, then the sequence $\{a_k\}$ is convergent. **True** **False**

(h) $\int_0^{+\infty} \cos x = 0$. **True** **False**

(i) Distance traveled can be less than displacement. **True** **False**

(j) The integral $\int_1^5 \frac{1}{(x-3)^2}\,dx$ is improper. **True** **False**

2. (20 pts) Evaluate

(a) $\int_0^{\pi/4} \sqrt{\tan x}\sec^2 x\,dx = \int_0^1 \sqrt{w}\,dw =$

$w = \tan x$
$dw = \sec^2 x\,dx$

$= \int_0^1 w^{\frac{1}{2}}dw = \frac{2}{3}w^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{2}{3}}$

(b) $\int x^2 \ln x\,dx \overset{I.B.P.}{=}$

$du = x^2 dx \qquad v = \ln x$
$u = \frac{x^3}{3} \qquad dv = \frac{1}{x}dx$

$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}dx =$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$

8. (20 pts) (a) (12 pts) Find the interval of convergence (with endpoints) of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \cdot 3^{k-1}}(x-5)^k$.

(b) (8 pts) Determine a function $f(x)$ whose Taylor series at $x_0 = 5$ is the series in part (a). Hint: Find first $f'(x)$.

(a) Abs. Ratio Test $\rightarrow$ $\boxed{I = (2, 8]}$ interval of convergence

$$\rho = \lim_{k\to\infty} \frac{|x-5|^{k+1}}{(k+1)\cdot 3^{k}} \cdot \frac{k\cdot 3^{k-1}}{|x-5|^{k}} = \lim_{k\to\infty}\left(\frac{k}{k+1}\cdot\frac{|x-5|}{3}\right) = \frac{|x-5|}{3}$$

If $\rho = \frac{|x-5|}{3} < 1$ $\rightarrow$ series is abs. convergent

If $\rho = \frac{|x-5|}{3} > 1$ $\rightarrow$ series is divergent

$$\frac{|x-5|}{3} < 1 \iff |x-5| < 3 \iff -3 < x-5 < 3 \iff 2 < x < 8$$

Test end-points: $x = 8$ $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k\cdot 3^{k-1}}\cdot 3^k = 3\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$ converges by AST

alternating harmonic

$x = 2 \rightarrow \sum \frac{(-1)^{k-1}(-3)^k}{k\cdot 3^{k-1}} = -\sum \frac{3}{k}$ $\leftarrow$ harmonic divergent

(b) $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k\cdot 3^{k-1}}\cdot(x-5)^k \Rightarrow f'(x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\cancel{k}\cdot 3^{k-1}}\cdot \cancel{k}(x-5)^{k-1}$

$$\text{so } f'(x) = \sum_{k=1}^{\infty}\left(-\frac{x-5}{3}\right)^{k-1} = \frac{1}{1+\frac{x-5}{3}} = \frac{3}{x-2}$$

geometric series

Thus $f(x) = \int \frac{3}{x-2}\,dx = 3\ln(x-2) + c$

c is determined from $f(5) = 0$ (since the series above starts from $k=1$)

thus $0 = 3\ln 3 + c \Rightarrow c = -3\ln 3$

so $f(x) = 3\ln(x-2) - 3\ln 3 = 3\ln\left(\frac{x-2}{3}\right)$

3. (20 pts) Evaluate

(a) $\displaystyle\int \frac{1}{x(2x+1)}\,dx$

partial fractions

$$\frac{1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

Guessing, or with algebra, {or both}

$A = 2$, $B = -1$

Thus

$$\int \frac{1}{x(2x+1)}\,dx = \int\left(\frac{2}{x} - \frac{1}{2x+1}\right)dx$$

$$= 2\ln x - \frac{1}{2}\ln(2x+1) + C$$

(b) $\displaystyle\int \frac{1}{(25-x^2)^{3/2}}\,dx \quad = *$

trig. sub. $x = 5\sin\theta$

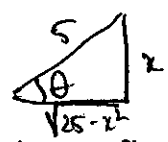
$dx = 5\cos\theta\,d\theta$

$$(25-x^2)^{\frac{3}{2}} = (25 - 25\sin^2\theta)^{\frac{3}{2}} =$$

$$= (25\cos^2\theta)^{\frac{3}{2}} = (5\cos\theta)^3 = 5^3\cdot\cos^3\theta$$

$$* = \int \frac{5\cos\theta\,d\theta}{5^3\cos^3\theta} = \frac{1}{25}\int\frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{25}\int \sec^2\theta\,d\theta = \frac{1}{25}\tan\theta + C$$



$$\boxed{= \frac{1}{25}\cdot\frac{x}{\sqrt{25-x^2}} + C}$$

$\sin\theta = \frac{x}{5} \Rightarrow \tan\theta = \frac{x}{\sqrt{25-x^2}}$

4. Choose ONE. Note the different point values.

(a) (15 pts) State and prove the Geometric Series Theorem.

(b) (10 pts) State and prove the integration by parts formula.

see notes or text

9. (16 pts) (a) (10 pts) The function $f(x) = \frac{1}{2}(e^x + e^{-x})$ is called the hyperbolic cosine of x, and is traditionally denoted $\cosh x$. Find its MacLaurin series. Your final answer should use summation notation.

(b) (6 pts) Based on part (a), one writes the approximation $\cosh(0.2) \approx 1 + \frac{(0.2)^2}{2}$. Can you guarantee that the error in this approximation is less than 0.001? Explain your answer.
Hint: Recall that $|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$ and also note that, for $\cosh x$, the MacLaurin polynomials of degrees 2 and 3 coincide.

(a) You may do the problem using the definition, but easiest is to use the MacLaurin series of e^x that you know already

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Thus

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

(b) The MacLaurin polys of deg 2 & 3 of $\cosh x$ are both $1 + \frac{x^2}{2!}$. Thus $\cosh x \approx 1 + \frac{x^2}{2!}$, but for the error estimate we can use $\underline{n=3}$.

$$f^{(n+1)}(x) = f^{(4)}(x) = \frac{1}{2}(e^x + e^{-x})$$

if $x \in [0, 0.2]$ $\quad |f^{(n+1)}(x)| < \frac{1}{2}e^{0.2} + \frac{1}{2}e^0 < \frac{3}{2} + \frac{1}{2} = 2$

so we can take $M = 2$ ($M=3$ also OK, obviously but $M=1$ would not be OK)

$$\text{so } |R_3(0.2)| \leq \frac{2 \cdot (0.2)^4}{4!} = \frac{2 \cdot \left(\frac{1}{5}\right)^4}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{1}{5^4 \cdot 3 \cdot 4} = \frac{1}{625 \times 3 \times 4} \underset{\text{true}}{<} \frac{1}{1000}$$

So, yes we can guarantee that the error is less than 0.001.

6. (24 pts) Determine whether each of the following series is absolutely convergent (AC), conditionally convergent (CC), or divergent. Be sure to state which test you are using and to show that it applies to the series in question.

(a) $\displaystyle\sum_{k=1}^{+\infty}(-1)^{k-1}\frac{1}{k^2+1}$

$\boxed{\text{AC because}}$

$$\sum_{k=1}^{+\infty}\left|(-1)^{k-1}\frac{1}{k^2+1}\right| = \sum_{k=1}^{\infty}\frac{1}{k^2+1}$$

and $\dfrac{1}{k^2+1} < \dfrac{1}{k^2}$

and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ convergent p-series $p=2>1$

so $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2+1}$ is conv. by simple comparison test

(b) $\displaystyle\sum_{k=2}^{+\infty}(-1)^{k}\frac{\ln k}{k}$

It is $\underline{\text{not AC}}$ since

$$\sum_{k=2}^{\infty}\left|(-1)^k\frac{\ln k}{k}\right| = \sum_{k=2}^{\infty}\frac{\ln k}{k} \geq \underbrace{\sum_{k=2}^{\infty}\frac{1}{k}}_{\text{harmonic}} = +\infty$$

To check for CC we need to see if A.S.T. applies

let $a_k = \dfrac{\ln k}{k}$

$$\lim_{k\to+\infty} a_k = \lim_{x\to+\infty}\frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}} \lim_{x\to+\infty}\frac{\frac{1}{x}}{1} = 0$$

We also need to check that $\{a_k\}$ is eventually decreasing.

If $f(x) = \dfrac{\ln x}{x}$, $f'(x) = \dfrac{\frac{1}{x}\cdot x - (\ln x)\cdot 1}{x^2}$

$$f'(x) = \frac{1-\ln x}{x^2} < 0 \text{ if } x>e$$

Thus $f(x)$ is decreasing when $x>e$

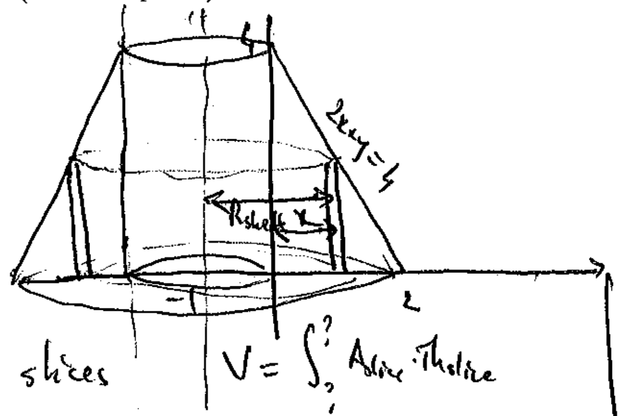
so $a_k = f(k)$ is decreasing for $k \geq 4$.

Thus, A.S.T. applies so $\displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k}$ is convergent

$\boxed{\text{thus, } \displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k} \text{ is C.C.}}$

5. (30 pts) Sketch a picture and then set up an integral representing each of the following. Computation of the integral is **not** required for this problem.

(a) Volume of the solid obtained by revolving the region bounded by $x = 0$, $2x + y = 4$ and $y = 0$ around the line $x = -1$ (sketch required).



Sol. with cyl. shells.

$V = \int_?^? 2\pi R_{shell} \cdot h_{shell} \cdot Th_{shell}$

$Th_{shell} = dx$, $R_{shell} = x+1$, $h_{shell} = y = 4-2x$

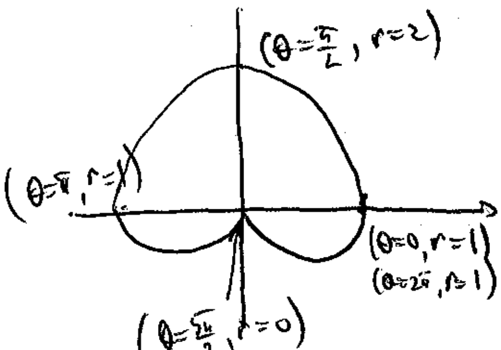
$$V = 2\pi \int_{x=0}^{x=2} (x+1)(4-2x)\,dx$$

Sol. with slices

$V = \int_?^? A_{slice} \cdot Th_{slice}$

$$V = \int_{y=0}^{y=4} \pi\left[\left(2-\frac{y}{2}+1\right)^2 - 1^2\right] dy = \pi \int_0^4 \left[\left(3-\frac{y}{2}\right)^2 - 1\right] dy$$

(b) Area of the region inside the cardioid $r = 1 + \sin\theta$ and below the x-axis. Sketch required.

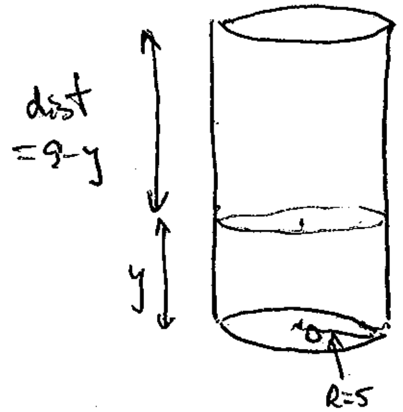


$$A = \frac{1}{2}\int_{\theta=\pi}^{\theta=2\pi} (1+\sin\theta)^2\, d\theta$$

or, by symmetry,

$$A = \cancel{2} \cdot \frac{1}{\cancel{2}} \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} (1+\sin\theta)^2\, d\theta$$

(c) Work. A vertical cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with gasoline of density $\rho = 50$ lbs/ft^3. Set up an integral for the work required to pump all the gasoline over the upper rim.



My choice of origin is at the bottom of the tank and let y be the height of the gas remaining

$$W = \int_?^? \rho \cdot A_{slice} \cdot Th_{slice} \cdot dist = \boxed{\int_{y=0}^{y=6} 50 \cdot 25\pi \cdot (9-y)\, dy}$$

$Th_{slice} = dy$

$dist = 9 - y$

$A_{slice} = \pi R^2 = \pi \cdot 5^2 = 25\pi$

since radius of slice is constant 5