Name: Solution Key                    **Panther ID:** ____________

**FINAL EXAM**        Calculus II        Spring 2014

**Important Rules:**

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

**1.** (15 pts) Circle the correct answer. No justification is necessary for this problem.

**(a)** The arc length of $y = e^{2x}$ from $x = 2$ to $x = 4$ is given by

(i) $\dfrac{e^8 - e^4}{4-2}$    (ii) $\displaystyle\int_2^4 e^{2x}\,dx$    (iii) $\displaystyle\int_2^4 (1+e^{2x})\,dx$    (iv) $\displaystyle\int_2^4 \sqrt{1+4e^{4x}}\,dx$    (v) $e^6$

**(b)** The expression

$$\frac{d}{dx}\left(\int_1^{x^3} \frac{\sin t}{t}\,dt\right) \text{ is equivalent to}$$

(i) $\dfrac{\sin(x^3)}{x^3}$    (ii) $\sin 1$    (iii) $\dfrac{3\sin(x^3)}{x}$    (iv) $3x^2\dfrac{\sin t}{t}$    (v) $0$

**(c)** Let $f(x)$ be an increasing continuous function on $[a, b]$ and let $R_4$ be the right end point Riemann sum approximation with 4 subdivisions of the integral $\int_a^b f(x)\,dx$. Then compared with the integral, $R_4$ is an

(i) overestimate    (ii) underestimate    (iii) exact estimate    (iv) cannot tell (more should be known about $f$)

**(d)** The sequence $a_n = 2+(-1)^n$, $n \geq 1$ is

(i) convergent but not monotone    (ii) monotone but divergent    (iii) bounded but divergent
(iv) eventually decreasing but unbounded    (v) none of the above

**(e)** The average value of the function $f(x) = x^3$ over the interval $[0, 1]$ is

(i) $\frac{1}{2}$    (ii) $3x^2$    (iii) $\frac{1}{4}$    (iv) $\frac{1}{8}$    (v) none of these

**2.** (10 pts) Find the total distance traveled by a particle which moves along the $x$ axis with a velocity $v(t) = 2t - 4$ meters/second, for $0 \le t \le 6$ seconds.

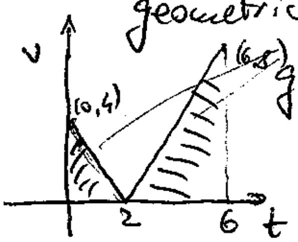
$$\text{Total distance} = \int_0^6 |v(t)|\,dt = \int_0^6 |2t-4|\,dt =$$

$$= \int_0^2 -(2t-4)\,dt + \int_2^6 (2t-4)\,dt =$$

$$= (4t-t^2)\Big|_0^2 + (t^2-4t)\Big|_{t=2}^{t=6}$$

$$= 4 + 16 = 20 \text{ meters}$$

Note: Problem can be done geometrically, as well

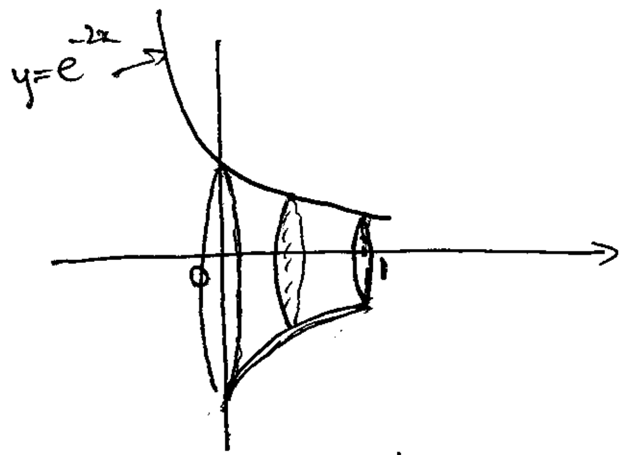



graph of $|v(t)|$

Total dist = shaded a

$$= \frac{4\times 2}{2} + \frac{8\times 4}{2} = 20$$

**3.** (14 pts) Find the volume of the solid that results when the region enclosed by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = 1$ is revolved about the $x$-axis. (Computation is required. Sketch of solid is also required.)




Cross-section method

$$V = \int_?^? A_{slice} \cdot Th$$

$$Th = dx$$

$$A_{slice} = \pi R^2 = \pi (e^{-2x})^2 = \pi e^{-4x}$$

$$R = e^{-2x}$$

$$\text{Thus } V = \int_0^1 \pi e^{-4x}\,dx = -\frac{\pi}{4} e^{-4x}\Big|_{x=0}^{x=1} = -\frac{\pi}{4}(e^{-4} - e^{0})$$

$$\boxed{V = \frac{\pi}{4}(1 - e^{-4})}$$

4. (10 pts) Evaluate the improper integral or show is divergent $\int_0^{+\infty} \frac{x}{x^2+1}\, dx$ Diverges

$$\int \frac{x}{x^2+1}\,dx = \int \frac{\frac{1}{2}\,dw}{w} = \frac{1}{2}\ln w = \frac{1}{2}\ln(x^2+1)+c$$

$w = x^2+1$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$$\int_0^{+\infty}\frac{x}{x^2+1}dx = \lim_{b\to+\infty}\int_0^b \frac{x}{x^2+1}dx = \lim_{b\to+\infty}\left(\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 1\right) = +\infty$$

5. (20 pts) Evaluate (10 pts each):

(a) $\int x \ln x\, dx$ =

I.B.P.

$du = x\,dx$ $\quad v = \ln x$

$u = \frac{x^2}{2}$ $\quad dv = \frac{1}{x}dx$

$$= \frac{x^2}{2}\ln x - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}dx$$

$$= \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + c}$$

(b) $\int \frac{1}{\sqrt{4+x^2}}\, dx$ =

Trig sub: $x = 2\tan\theta$

$\sqrt{4+x^2} = \sqrt{4(1+\tan^2\theta)} = 2\sec\theta$

$dx = 2\sec^2\theta\, d\theta$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{2\sec\theta}} = \int \sec\theta\, d\theta =$$

$$= \ln(\sec\theta + \tan\theta) + c$$

$$\boxed{= \ln\left(\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right) + c}$$

$$= \ln\left(\frac{x+\sqrt{4+x^2}}{2}\right) + c$$

$$= \ln\left(x+\sqrt{4+x^2}\right) + c - \ln 2$$

$$= \ln\left(x+\sqrt{4+x^2}\right) + \tilde{c}$$

**6.** (14 pts) Determine if the series $\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)}$ converges. If so, find the sum of the series.

Realize the series is telescopic.
Indeed, with guess & adjust (or rigorous partial fractions) get

$$\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$$

Thus $S_n = \sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \ldots + \left(\cancel{\frac{1}{2k-1}}\right.$

$$S_n = 1 - \frac{1}{2n+1}$$

Thus $\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)} = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{2n+1}\right) = 1$. Series converges to $1$.

**7.** (12 pts) Determine if $\sum_{k=2}^{\infty} \frac{(-1)^k}{k(\ln k)^2}$ is absolutely convergent, conditionally convergent, or divergent. Justify.

Look at the series of absolute values

$$\sum_{k=2}^{\infty} \left|\frac{(-1)^k}{k(\ln k)^2}\right| = \sum_{k=2}^{+\infty} \frac{1}{k(\ln k)^2}$$

To this we apply the integral test

$f(x) = \frac{1}{x(\ln x)^2}$ is decreasing on $[2, +\infty)$

$$\int_2^{+\infty} \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 2}^{+\infty} \frac{1}{w^2}\,dw = -\frac{1}{w}\Bigg|_{w=\ln 2}^{w=+\infty}$$

$w = \ln x$

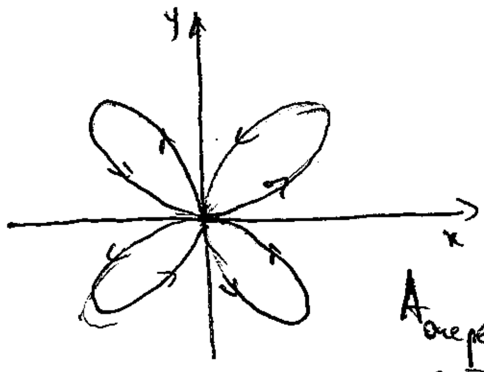
$dw = \frac{1}{x}dx$

$$= 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

As the integral converges, $\sum_{k=2}^{+\infty} \frac{1}{k(\ln k)^2}$ is convergent,

hence $\sum_{k=2}^{\infty} \frac{(-1)^k}{k(\ln k)^2}$ is absolutely convergent

**8.** (12 pts) Sketch the rose $r = \sin(2\theta)$ and compute the area of one petal.




The petal in 1st quadrant starts at $\theta = 0$ (since $r = \sin(2 \cdot 0) = 0$)

and ends ~~when~~ when $\theta = \frac{\pi}{2}$

(since $r = \sin(2 \cdot \frac{\pi}{2}) = \sin\pi = 0$ again)

$$A_{\text{one petal}} = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta =$$

$$\underset{\substack{\uparrow \\ \text{double angle} \\ \text{formula}}}{=} \frac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1 - \cos(4\theta)}{2} \, d\theta = \frac{1}{4} \left( \theta - \frac{\sin(4\theta)}{4} \right) \Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \frac{1}{4} \cdot \frac{\pi}{2} = \boxed{\frac{\pi}{8}}$$

**9.** (14 pts) Choose ONE:
(a) State and prove FTC part (b) (the one about $\frac{d}{dx} \int$).
(b) State and prove the the integration formula for area in polar coordinates. A picture, a sum and a limit should appear in your work.

See text or notes

**10.** (6 pts) Write the partial fraction decomposition. It is NOT required to determine the constants.

$$\frac{1}{(x+2)^3(x^2+4)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

**11.** (14 pts) Find the radius and the interval of convergence for $\sum_{k=1}^{\infty} \frac{(-1)^k (x-1)^k}{3^k \sqrt[3]{k}}$

Absolute Ratio Test

$$\rho = \lim_{k\to\infty} \frac{|x-1|^{\cancel{k+1}}}{3^{\cancel{k+1}} \sqrt[3]{k+1}} \cdot \frac{\cancel{3^k} \cdot \sqrt[3]{k}}{\cancel{|x-1|^k}} = \lim_{k\to\infty} \left( \frac{|x-1|}{3} \cdot \frac{\sqrt[3]{k}}{\sqrt[3]{k+1}} \right) = \frac{|x-1|}{3}$$

$$\rho < 1 \iff \frac{|x-1|}{3} < 1 \iff |x-1| < 3$$

Thus radius of convergence is $\boxed{R=3}$

$$-3 < x-1 < 3 \iff -2 < x < 4$$

Test the end-points

$x=-2$ $$\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (-3)^k}{3^k \cdot \sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{(-1)^{2k} \cancel{3^k}}{\cancel{3^k} \cdot \sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{3}}} \text{ divergent}$$

p-series with $p = \frac{1}{3} < 1$

$x = 4$ $$\sum_{k=1}^{\infty} \frac{(-1)^k \cdot \cancel{3^k}}{\cancel{3^k} \cdot \sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}} \text{ convergent by A.S.T.}$$

$a_k = \frac{1}{\sqrt[3]{k}} \searrow 0$, so A.S.T. applies.

**12.** (14 pts) (a) (6 pts) Write the Maclaurin series for $e^x$ and use it to find a series whose sum is $1/\sqrt{e}$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \cdot k!}$$

(b) (8 pts) What is the smallest $n$ so that the partial sum $S_n$ of the series in part (a) approximates $1/\sqrt{e}$ with an error less than $10^{-3}$? Be sure to justify your answer.

Can apply the alt. series error estimate (Same ~~estimate~~ value for n would be obt with the Remainder Estimate

$$\left| \frac{1}{\sqrt{e}} - S_n \right| \leq \frac{1}{2^{n+1} \cdot (n+1)!}$$

Want to find the smallest $n$ so that $\frac{1}{2^{n+1} \cdot (n+1)!} \leq \frac{1}{10^3}$

If $n=3$ $\quad 2^{n+1} \cdot (n+1)! = 2^4 \cdot 4! = 16 \cdot 24 < 1000$ (not good enough!)

If $n=4$ $\quad 2^{n+1} \cdot (n+1)! = 2^5 \cdot 5! = 32 \cdot 120 > 1000$

so $\frac{1}{2^5 \cdot 5!} < \frac{1}{1000}$

Thus the smallest $n$ is $n=4$. The approximation is $\frac{1}{\sqrt{e}} \approx 1 - \frac{1}{2 \cdot 1!} + \frac{1}{2^2 \cdot 2!} - \frac{1}{2^3 \cdot 3!} + \frac{1}{2^4 \cdot 4!}$

**13.** (10 pts) Differentiate a familiar series to obtain the MacLaurin series for $\frac{1}{(1-x)^2}$.

As $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $-1 < x < 1$

Differentiate both sides and get

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k\, x^{k-1} = \sum_{\ell=0}^{\infty} (\ell+1)\, x^{\ell}$$

$\ell = k-1$