Name: Solution Key Panther ID: __________

Exam 1 Calculus II Spring 2019

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (8 pts) Find the average value of $f(x) = \sec^2 x$ on the interval $[0, \pi/4]$.

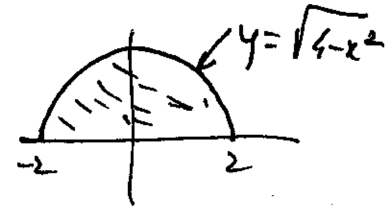
$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$, so in this case

(or $av(f)$, as in the text)

$$f_{ave} = \frac{1}{\frac{\pi}{4}-0}\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \frac{4}{\pi}\cdot(\tan x)\Big|_{x=0}^{x=\frac{\pi}{4}} = \frac{4}{\pi}(1-0) = \boxed{\frac{4}{\pi}}$$

2. (8 pts) Find $\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi\cdot 2^2 = \boxed{2\pi}$

Geometrically, the integral represents the area of a semicircle of radius 2.



3. (8 pts) Find $\frac{d}{dx}\int_0^{\sqrt{x}} \cos(t^2)\,dt = \cos\left((\sqrt{x})^2\right)\cdot(\sqrt{x})' = \boxed{\frac{\cos x}{2\sqrt{x}}}$

using FTC & Chain Rule

4. (8 pts) Find $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$ $= \arctan x \Big|_{x=-1}^{x=1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$

5. (8 pts) Find $\int_{1}^{e} \frac{(\ln x)^3}{x}\,dx$ $\overset{\uparrow}{=} \int_{u=0}^{u=1} u^3\,du = \frac{u^4}{4}\Big|_{u=0}^{u=1} = \boxed{\frac{1}{4}}$

sub $u = \ln x$

$du = \frac{1}{x}\,dx$

6. (8 pts) Find $\int_{0}^{1} x\sqrt{1+3x}\,dx$ $= \ast$

sub $w = 1+3x \Rightarrow x = \frac{w-1}{3}$ $\qquad x=0 \Rightarrow w=1$

$dw = 3\,dx$ $\qquad x=1 \Rightarrow w=4$

$\frac{1}{3}dw = dx$

$$\ast = \int_{w=1}^{w=4} \frac{w-1}{3}\sqrt{w}\cdot\frac{1}{3}dw = \frac{1}{9}\int_{w=1}^{w=4}(w-1)w^{\frac{1}{2}}\,dw = \frac{1}{9}\int_{w=1}^{w=4}\left(w^{\frac{3}{2}} - w^{\frac{1}{2}}\right)dw$$

$$= \frac{1}{9}\left(\frac{2}{5}w^{\frac{5}{2}} - \frac{2}{3}w^{\frac{3}{2}}\right)\Big|_{w=1}^{w=4} = \frac{1}{9}\left(\frac{2}{5}4^{\frac{5}{2}} - \frac{2}{3}4^{\frac{3}{2}}\right) - \frac{1}{9}\left(\frac{2}{5}\cdot1 - \frac{2}{3}\cdot1\right)$$

7. (8 pts) Use summation notation and then find the exact value of the sum:

$1+3+5+7+ \ldots +2017+2019 \quad = \sum_{k=1}^{1010}(2k-1) = 2\sum_{k=1}^{1010} k - \sum_{k=1}^{1010} 1 =$

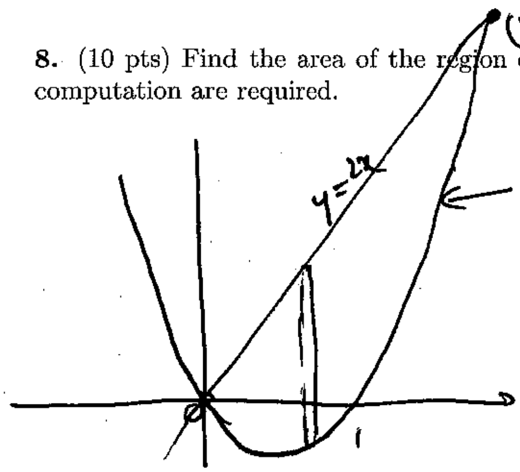
It's OK to leave your answer as a product!

$$= 2 \cdot \frac{1010 \cdot 1011}{2} - 1010 =$$

Gauss's formula

$$= 1010 \cdot (1011 - 1) = \boxed{1010^2}$$

8. (10 pts) Find the area of the region enclosed by the parabola $y = x^2 - x$ and the line $y = 2x$. Sketch and computation are required.



$\leftarrow y=x^2-x$ Intersection points of $y=x^2-x$ and $y=2x$

$$\begin{cases} y = x^2 - x \\ y = 2x \end{cases} \Rightarrow x^2 - x = 2x \Rightarrow x^2 - 3x = 0$$

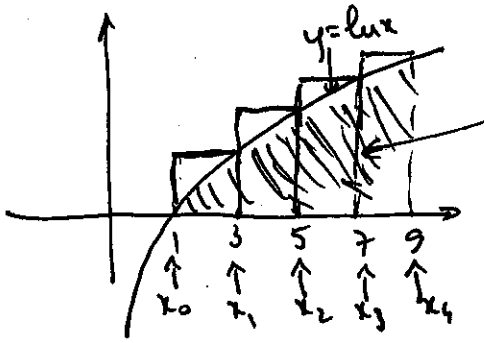
$$\Rightarrow x=0,\ x=3$$

$$\text{Area} = \int_0^3 (2x-(x^2-x))\,dx = \int_0^3 (3x-x^2)\,dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=3} = \frac{3 \cdot 3^2}{2} - \frac{3^3}{3} = 27\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= 27 \cdot \frac{1}{6} = \boxed{\frac{9}{2} = 4.5}$$

9. (10 pts) (a) (3 pts) Sketch a graph of $y = \ln x$ and on this graph shade an area corresponding to $\int_1^9 \ln x\, dx$.



$\int_1^9 \ln x\, dx$ (the area shaded "≡")

(b) (3 pts) On your graph from part (a) or on a new graph, next mark the area corresponding to R_4^{right}, the right-endpoint Riemann sum approximation with 4 equal subdivisions of $\int_1^9 \ln x\, dx$.

R_4^{right} is the sum of the areas of the rectangles indicated above

(c) (1 pts) Is R_4^{right} an over-estimate or an under-estimate of the integral?

It is an over-estimate of the integral

(d) (3 pts) Write the concrete expression corresponding to R_4^{right}. You don't have to simplify, but your answer should be in a calculator ready form.

$$R_4^{right} = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \Delta x \cdot f(x_4)$$

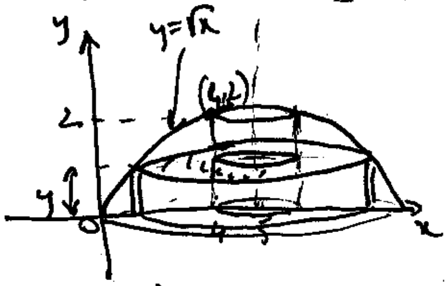
so $R_4^{right} = 2 \cdot \ln 3 + 2 \cdot \ln 5 + 2 \cdot \ln 7 + 2 \cdot \ln 9$

(combining the logarithms, this can also be written as

$$R_4^{right} = 2 \cdot \ln(3 \cdot 5 \cdot 7 \cdot 9)$$

10. (8+4 pts) Set up an integral (or integrals) to represent the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 4$ around the line $x = 5$. Be sure to indicate which method you are using, cross-section or cylindrical shells. Computation is **not** required, but a picture is.
Bonus: Up to 4 bonus points if you correctly solve this problem both ways.

Cross-section method



$$V = \int_?^? A_{slice} \cdot Th_{slice}$$

$Th_{slice} = dy$

$A_{slice} = \pi R^2 - \pi r^2$

$R = 5 - x_{curve} = 5 - y^2$

$r = 5 - 4 = 1$

$$V = \int_{y=0}^{y=2} \pi\left((5-y^2)^2 - 1\right) dy$$

Cylindrical shells method

$$V = \int_?^? 2\pi R_{shell} h_{shell} \cdot Th_{shell}$$

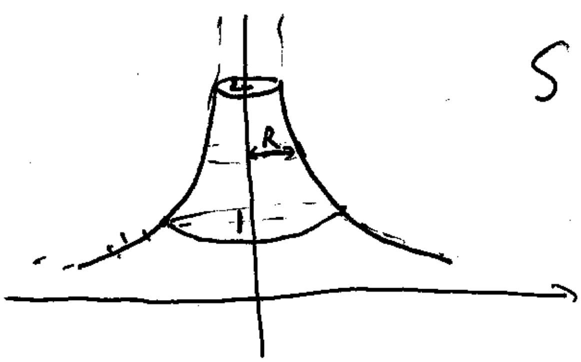
$Th_{shell} = dx$

$h_{shell} = y_{curve} = \sqrt{x}$

$R_{shell} = 5 - x$ ← or like this.

$$V = 2\pi \int_{x=0}^{x=4} (5-x)\sqrt{x}\, dx$$

11. (8 pts) Set up an integral (or integrals) to represent the surface area generated by revolving the graph of $xy = 1$, $1 \le y \le 2$, around the y-axis. (Again, just the set-up of the integral is required, not the computation).



$$S = \int_?^? 2\pi R_{strip}\, ds$$

$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

(but ok if you work with $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ and are consistent)

$R_{strip} = x_{curve} = \frac{1}{y}$ (as we chose dy to factor out in ds)

$$S = \int_{y=1}^{y=2} 2\pi \frac{1}{y} \cdot \sqrt{1 + \left(-\frac{1}{y^2}\right)^2}\, dy = \underbrace{2\pi \int_{y=1}^{y=2} \frac{1}{y}\sqrt{1 + \frac{1}{y^4}}\, dy}_{\text{ok, like this.}}$$

12. (10 pts) Choose ONE. If you do both, only the larger score will be considered for this problem, but the second score may give some bonus towards a previous problem where your score is smaller.

(A) State and prove the part of FTC about $\frac{d}{dx}(\int_a^x \ldots)$. You may use without proof MVT for integrals.

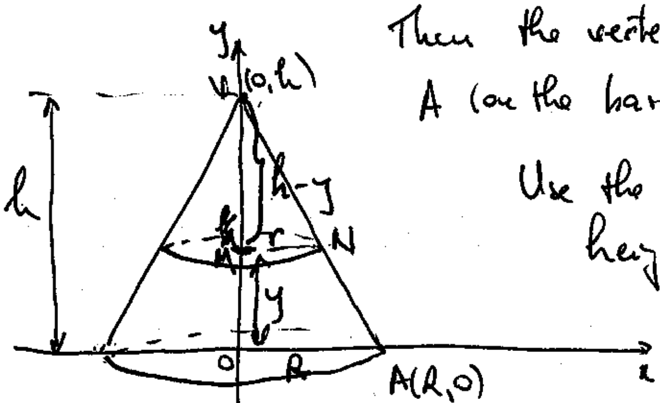
(B) Use integrals to show that the volume of a cone is given by $V = \frac{1}{3}h \cdot A_{base}$, where h denotes the height of the cone (the distance from the vertex to the base) and A_{base} denotes the area of the base. For simplicity, it's OK to consider just the case of a right circular cone, although the formula is valid for any cone (and any pyramid for that matter).

See notes of text for (A)

(B) Here is the argument for the right circular cone (ask me in class about the general case)

Choose a coordinate system that has the origin at the center of the base

Then the vertex, V, has coordinates $(0, h)$ and the point A (on the base) coordinates $(R, 0)$, where R is the radius of the base.



Use the slicing method and consider a slice at a height y from the base

$$V_{cone} = \int_{y=?}^{?} A_{slice} \cdot Th_{slice}$$

$Th_{slice} = dy$

$A_{slice} = \pi r^2$

To get 'r' in terms of y use the similarity of ΔVMN and ΔVOA

Then $\frac{r}{R} = \frac{h-y}{h}$ so $r = \frac{(h-y)R}{h}$

$$\int (h-y)^2 dy = -\frac{1}{3}(h-y)^3$$

Thus
$$V_{cone} = \int_{y=0}^{y=h} \pi \cdot \frac{(h-y)^2 R^2}{h^2} dy = \pi \frac{R^2}{h^2} \int_{y=0}^{y=h} (h-y)^2 dy =$$

$$= \pi \frac{R^2}{h^2} \cdot \left(-\frac{1}{3}\right)(h-y)^3 \Big|_{y=0}^{y=h} = \pi \frac{R^2}{h^2}\left(0 - \left(-\frac{1}{3}\right) h^3\right)$$

$$= \frac{\pi R^2}{h^2} \cdot \frac{1}{3} h^3 = \boxed{\frac{1}{3} h \cdot \pi R^2}$$ as claimed