Name: Solution Key Panther ID: ____________

Exam 2 Calculus II Spring 2019

To receive credit you MUST SHOW ALL YOUR WORK.

1. (12 pts) Circle the correct answer (3 pts each):

(a) For the integral $\int \sqrt{9x^2+4}\,dx$, the following substitution is helpful:

(i) $x = \tan\theta$ (ii) $3x = 2\sin\theta$ (iii) $x = 3\sec\theta$ (iv) $3x = 2\tan\theta$ (v) $w = 9x^2+4$

(Don't spend time evaluating the integral. It is not required.)

(b) The partial fraction decomposition for $\frac{x+3}{(x+2)^2(x^2+4)}$ is of the form:

(i) $\dfrac{A}{x+2} + \dfrac{B}{x^2+4}$ (ii) $\dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{Cx+D}{x^2+4}$ (iii) $\dfrac{x+3}{(x+2)^2} + \dfrac{x+3}{x^2+4}$

(iv) $\dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3} + \dfrac{D}{(x+2)^4}$ (v) none of the above

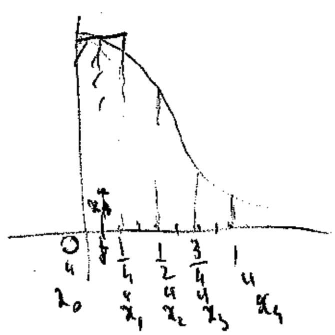
(c) A function $f(x)$ is known to be continuous, positive and concave down when $x \in [-2,2]$. Let T_4 be the trapezoid approximation with 4 subdivisions of the integral $\int_{-2}^{2} f(x)\,dx$. Then compared with the integral, T_4 is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about f)

(d) A function $f(x)$ is known to be continuous, positive and concave down when $x \in [-2,2]$. Let R_4^{right} be the right end-point approximation with 4 subdivisions of the integral $\int_{-2}^{2} f(x)\,dx$. Then compared with the integral, R_4^{right} is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about f)

2. (8 pts) Write an expression corresponding to M_4, the midpoint approximation with 4 subdivisions, for the integral $\int_0^1 e^{-t^2}\,dt$. Leave your answer in a calculator ready form, but you do not need to try to evaluate.



$M_4 = \Delta x\left(f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)\right)$

where x_k^* is the midpoint of interval $[x_{k-1}, x_k]$

In this case, $\Delta x = \frac{1}{4}$

$x_1^* = \frac{1}{8}$, $x_2^* = \frac{\frac14+\frac12}{2} = \frac{3}{8}$, $x_3^* = \frac{5}{8}$, $x_4^* = \frac{7}{8}$

$$M_4 = \frac{1}{4}\left(e^{-(\frac18)^2} + e^{-(\frac38)^2} + e^{-(\frac58)^2} + e^{-(\frac78)^2}\right)$$

For Problems 3-6, evaluate each integral.

3. (10 pts) $\int_0^1 \frac{x}{4x^2+1}\,dx$ $= \int_{w=1}^{w=5} \frac{\frac{1}{8}dw}{w} = \frac{1}{8} \ln|w| \Big|_{w=1}^{w=5}$

substitution

$w = 4x^2+1$

$dw = 8x\,dx$

$\frac{1}{8}dw = x\,dx$

$= \frac{1}{8}(\ln 5 - \ln 1) = \boxed{\frac{1}{8}\ln 5}$

4. (10 pts) $\int x^2 e^{2x}\,dx$ $= \frac{1}{2}x^2e^{2x} - \int \frac{1}{2}e^{2x}\cdot 2x\,dx =$

Integration by parts

$du = e^{2x}dx \qquad v = x^2$

$u = \int e^{2x}dx = \frac{1}{2}e^{2x} \qquad dv = 2x\,dx$

$= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx\right) =$

One more time I.B.P. for $\int x e^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$

$du = e^{2x}dx \qquad v = x$

$u = \frac{1}{2}e^{2x} \qquad dv = dx$

$= \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c}$

or $= \frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + c$

use identity $\tan^2 x = \sec^2 x - 1$

5. (10 pts) $\int \tan^3 x \sec x\, dx = \int \tan^2 x \sec x \tan x \, dx =$

$$= \int (\sec^2 x - 1) \sec x \tan x \, dx$$

use sub. $w = \sec x$

$dw = \sec x \tan x \, dx$

$$= \int (w^2 - 1)\, dw =$$

$$= \frac{w^3}{3} - w + C = \boxed{\frac{\sec^3 x}{3} - \sec x + C}$$

6. (12 pts) $\int_0^2 \frac{x^3}{\sqrt{4-x^2}}\, dx = *$

Trig. sub. $x = 2\sin\theta$

$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

New limits of integration $x = 0 \Rightarrow 2\sin\theta = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow 2\sin\theta = 2 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$* = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{(2\sin\theta)^3}{\cancel{2\cos\theta}} \cdot \cancel{2\cos\theta}\, d\theta = 8\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^3\theta\, d\theta =$$

$$= 8\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta \cdot \sin\theta\, d\theta = 8\int_{\theta=0}^{\theta=\frac{\pi}{2}} (1-\cos^2\theta)\sin\theta\, d\theta =$$

sub. $w = \cos\theta$

$dw = -\sin\theta\, d\theta$

$$= 8\int_{w=1}^{w=0} (1-w^2)(-dw) = 8\int_{w=0}^{w=1} (1-w^2)\,dw = 8\left(w - \frac{w^3}{3}\right)\Big|_{w=0}^{w=1} = 8\cdot\frac{2}{3} = \boxed{\frac{16}{3}}$$

7. (14 pts) Use partial fractions (or any other method) to compute

$$\int \frac{x+2}{x(x^2+4)}\,dx$$

Solution 1 (straight with partial fractions)

$$\frac{x+2}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \quad \Big|\cdot x(x^2+4)$$

$$x+2 = A(x^2+4) + x(Bx+C)$$

$$x+2 = (A+B)x^2 + Cx + 4A \quad \text{so} \quad \begin{cases} A+B=0 \\ C=1 \\ 4A=2 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=1 \end{cases}$$

$$\text{so } \int \frac{x+2}{x(x^2+4)}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \underbrace{\int \frac{-\frac{1}{2}x+1}{x^2+4}\,dx}_{\text{(split ~~...~~ distribute numerator}} = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{x}{x^2+4} + \int \frac{1}{x^2+4}$$

force 4 out factor

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\cdot\underbrace{\frac{1}{2}\ln(x^2+4)}_{\text{Guess \& Adjust!}} + \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+4) + \underbrace{\frac{1}{4}\,2\,\arctan\left(\frac{x}{2}\right)}_{\text{another guess \& adjust}} + c$$

$$\boxed{= \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c}$$

Solution 2 (trig sub)

$$\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned} \quad \text{so} \quad \int \frac{x+2}{x(x^2+4)}\,dx = \int \frac{2\tan\theta+2}{2\tan\theta\,(\underbrace{4\tan^2\theta+4}_{4\sec^2\theta})}\,2\sec^2\theta\,d\theta =$$

$$= \int \frac{\cancel{2}(\tan\theta+1)\,\cancel{2\sec^2\theta}}{2\tan\theta\cdot\cancel{4\sec^2\theta}}\,d\theta = \frac{1}{2}\int \frac{\tan\theta+1}{\tan\theta}\,d\theta = \frac{1}{2}\int\left(1+\frac{1}{\tan\theta}\right)d\theta$$

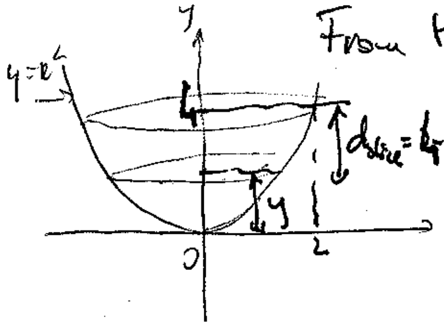
$$= \frac{1}{2}\int(1+\cot\theta)\,d\theta = \frac{1}{2}\left(\theta - \ln|\csc\theta|\right) + c$$

$\tan\theta = \frac{x}{2}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{\sqrt{x^2+4}}{x}$

$$\boxed{= \frac{1}{2}\left(\arctan\left(\frac{x}{2}\right) - \ln\left(\frac{\sqrt{x^2+4}}{x}\right)\right) + c}$$

8. (10 pts) The graph of $y = x^2$ for $0 \le x \le 2$ is revolved about the y-axis to form a tank that is then filled with salt water from the Dead Sea (weighing approximately 73 lb/ft^3). How much work does it take to pump all of the water to the top of the tank? (Assume both x and y are measured in feet.) Just set-up of the integral is required, you **DO NOT** have to evaluate the integral.



From the homework and done in class!

$$W = \int_{?}^{?} \rho \cdot A_{slice} \cdot dist_{slice} \cdot Th_{slice}$$

$Th_{slice} = dy$ $\qquad \rho = 73\ lb/ft^3$

$dist_{slice} = 4 - y$

$A_{slice} = \pi r^2 = \pi x^2 = \pi y$

$$\boxed{W = \int_{y=0}^{y=4} 73 \cdot \pi y \cdot (4-y)\, dy} = 73\pi \int_{y=0}^{y=4} (4y - y^2)\, dy$$

9. (10 pts) The following reduction formula holds for any positive constant a and any integers $m \ge 1$ and $n \ne -1$.

$$\int x^n (\ln ax)^m \, dx = \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} \, dx \qquad (*)$$

Use the above reduction formula to compute

$$\int x^3 (\ln 2x)^2 \, dx \underset{\uparrow}{=} \frac{x^4 (\ln 2x)^2}{4} - \frac{2}{4} \underbrace{\int x^3 (\ln 2x)^1 \, dx}_{} =$$

(*) with n=3, m=2, a=2 $\qquad\qquad$ (*) with n=3, m=1, a=2

$$= \frac{x^4(\ln 2x)^2}{4} - \frac{1}{2}\left[\frac{x^4 \ln 2x}{4} - \frac{1}{4}\int x^3\, dx\right] =$$

$$= \frac{x^4(\ln 2x)^2}{4} - \frac{1}{8} x^4 \ln 2x + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$= \frac{x^4(\ln 2x)^2}{4} - \frac{x^4(\ln 2x)}{8} + \frac{x^4}{32} + C$$

$$\text{or} = \frac{x^4}{4}\left((\ln 2x)^2 - \frac{1}{2}(\ln 2x) + \frac{1}{8}\right) + C$$

10. (10 pts) Choose ONE proof. If you do both, only the larger score will be considered for this problem, but the second score may give some bonus towards a previous problem where your score is smaller.

(A) State and prove the integration by parts formula.

(B) Prove the reduction formula stated in Problem 2:

$$\int x^n (\ln ax)^m \, dx = \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} \, dx$$

(A) → see notes or textbook

(B) Use integration by parts with

$du = x^n dx \qquad v = (\ln ax)^m$

chain Rule

$u = \dfrac{x^{n+1}}{n+1} \qquad dv = m(\ln ax)^{m-1} \cdot \dfrac{\cancel{a}}{\cancel{a}x} dx$

$$\int x^n (\ln ax)^m dx = \frac{x^{n+1} \cdot (\ln ax)^m}{n+1} - \int \frac{\cancel{x^{n+1}}^{\,x^n}}{n+1} \cdot m (\ln ax)^{m-1} \cdot \frac{1}{\cancel{x}} dx$$

So $$\int x^n (\ln ax)^m dx = \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx$$

which corresponds precisely to the given reduction formula