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Exam 3 Calculus II Spring 2019

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) The first six terms of a sequence $\{a_n\}_n$ are given below

$$a_1 = \frac{1}{3},\ a_2 = -\frac{2}{5},\ a_3 = \frac{3}{7},\ a_4 = -\frac{4}{9},\ a_5 = \frac{5}{11},\ a_6 = -\frac{6}{13},\ \ldots$$

(a) (6 pts) Assuming that the pattern continues, find the formula for the general term a_n.

$$a_n = (-1)^{n+1} \cdot \frac{n}{2n+1}$$

(b) (6 pts) Using your answer from (a), is the sequence $\{a_n\}_n$ convergent? Briefly justify your answer.

$\{a_n\}_n$ is divergent since it has subsequences with different limits

$$\lim_{k\to+\infty} a_{2k} = \lim_{k\to+\infty} -\frac{2k}{4k+1} = -\frac{1}{2} \neq \lim_{k\to+\infty} a_{2k+1} = \lim_{k\to+\infty} \frac{2k+1}{4k+3} = \frac{1}{2}$$

2. (12 pts) Evaluate each of the following (or show it diverges) (6 pts each)

(a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} \left[(\arctan x)\Big|_{x=0}^{x=b}\right]$

$$= \lim_{b\to+\infty} [\arctan b - \arctan 0] = \boxed{\frac{\pi}{2}}$$

(b) $\ln(1/3) + \ln(3/5) + \ln(5/7) + \ln(7/9) + \ldots = \displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{2k-1}{2k+1}\right)$ ← It's a telescopic series.

$$S_n = \sum_{k=1}^{n} \ln\left(\frac{2k-1}{2k+1}\right) = \sum_{k=1}^{n}\left(\ln(2k-1) - \ln(2k+1)\right)$$

$$S_n = \ln 1 - \cancel{\ln 3} + \cancel{\ln 3} - \cancel{\ln 5} + \cancel{\ln 5} - \cancel{\ln 7} + \ldots + \cancel{\ln(2n-1)} - \ln(2n+1)$$

$$S_n = \ln 1 - \ln(2n+1) = -\ln(2n+1)$$

Thus $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{2k-1}{2k+1}\right) = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(-\ln(2n+1)\right) = \boxed{-\infty}$

The series diverges to $-\infty$

3. (12 pts) Evaluate each of the following (or show it diverges) (6 pts each)

(a) $\lim_{k\to+\infty}\left(\frac{3}{2^k}-\frac{2}{3^k}\right)$ $= 0$ (as 2^k and 3^k go to $+\infty$ as $k\to+\infty$)

(b) $\sum_{k=0}^{+\infty}\left(\frac{3}{2^k}-\frac{2}{3^k}\right)$ $= 3\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^k - 2\cdot\sum_{k=0}^{+\infty}\left(\frac{1}{3}\right)^k =$

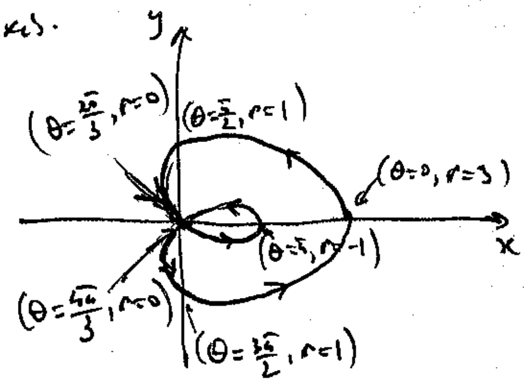
Both are convergent geometric series

$$= 3\cdot\frac{1}{1-\frac{1}{2}} - 2\cdot\frac{1}{1-\frac{1}{3}} = 6 - 2\cdot\frac{3}{2} = \boxed{3}$$

4. (12 pts) (a) (6 pts) Sketch the graph of the limacon with inner loop $r = 1 + 2\cos\theta$. Be sure to give the polar coordinates of all points where the graph intersects the x-axis, the y-axis, or passes through the origin.

symmetric w.r.t. the x-axis.

θ	r	
0	3	
$\frac{\pi}{2}$	1	
$\frac{2\pi}{3}$	0	← start of the inner loop
π	-1	
$\frac{4\pi}{3}$	0	← end of inner loop.



(b) (6 pts) Set up an integral that represents the area inside the inner loop of the limacon $r = 1 + 2\cos\theta$. Just set up of the integral is required, you **do NOT** have to compute the integral.

$$A = \frac{1}{2}\int_{\theta=\theta_1}^{\theta=\theta_2} r^2\, d\theta = \frac{1}{2}\int_{\theta=\frac{2\pi}{3}}^{\theta=\frac{4\pi}{3}} (1+2\cos\theta)^2\, d\theta$$

or, using symmetry

$$A = \frac{1}{\cancel{2}}\cdot\cancel{2}\int_{\theta=\frac{2\pi}{3}}^{\theta=\pi} (1+2\cos\theta)^2\, d\theta$$

5. (10 pts) In each case, answer True or False. No justification is required for this problem. (2 pts each)

(a) The graph of $r = 4\cos\theta$ in the Cartesian xy-plane is a circle that goes through the origin. **True** **False**

(b) If $a_k \geq \frac{1}{\sqrt{k}}$ for all $k \geq 1$, then $\sum_{k=1}^{\infty} a_k$ is divergent. **True** **False**

(c) If $\lim_{k\to+\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

(d) If $\lim_{k\to+\infty} \sqrt[k]{|a_k|} = \frac{4}{5}$ then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. **True** **False**

(e) Any bounded sequence is convergent. **True** **False**

6. (16 pts + 4 bonus pts) Determine whether each of the following series converges or diverges. Full justification is required. If you determine the precise value of the sum, you'll receive up to 4 bonus points.

(a) $\sum_{k=1}^{\infty} \frac{k}{2k+1}$

$\lim_{k\to+\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$ so series diverges by the k^{th}-term test

$\sum_{k=1}^{\infty} \frac{k}{2k+1} = +\infty$ (as is a divergent series of positive terms)

(b) $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$

observe the series is telescopic since $\frac{1}{4k^2-1} = \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$

(can get this by partial fractions or guess & adjust)

Then $S_n = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2n+1}\right)$ (all other terms cancel)

Thus $\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$

so the series converges to $\frac{1}{2}$.

Note: You could also apply a comparison test with the convergent p-series $\sum \frac{1}{k^2}$ and conclude that your series converges, but you'd not know its

7. (20 pts) For each of the following series, determine if the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Answer **and carefully** justify your answer. Very little credit will be given just for a guess. Most credit is given for the quality of the justification. (10 pts each)

(a) $\sum_{k=2}^{\infty}(-1)^k \frac{1}{k \ln k}$ ← Test absolute convergence $\sum \left| (-1)^k \frac{1}{k \ln k} \right| = \sum_{k=2}^{+\infty} \frac{1}{k \ln k}$

So the series is not AC

Apply integral test for this

$$\int_{x=2}^{x=+\infty} \frac{1}{x \ln x}\, dx \underset{\substack{u=\ln x \\ du=\frac{1}{x}dx}}{=} \int \frac{1}{u}\, du = \ln(\ln x)\Big|_{x=2}^{x=+\infty} = +\infty$$

...

We apply A.S.T.

As $u_k = \frac{1}{k \ln k}$ is decreasing

and $\lim_{k \to +\infty} \frac{1}{k \ln k} = 0$, the series $\sum_{k=2}^{\infty} (-1)^k \frac{1}{k(\ln k)}$ is convergent

Thus, $\sum_{k=2}^{+\infty} (-1)^k \frac{1}{k \ln k}$ is conditionally convergent (C.C.)

(b) $\sum_{k=0}^{\infty}(-1)^k \frac{(k!)^2}{(2k+1)!}$ Apply the Ratio Test

$$\rho = \lim_{k \to +\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \to +\infty} \frac{((k+1)!)^2}{(2(k+1)+1)!} \cdot \frac{(2k+1)!}{(k!)^2}$$

$$\rho = \lim_{k \to +\infty} \frac{(k+1)\cancel{k!} \cdot (k+1)\cancel{k!}}{\underbrace{\cancel{(2k+3)!}}_{(2k+3)(2k+2)}} \cdot \frac{\cancel{(2k+1)!}}{\cancel{k!} \cdot \cancel{k!}} = \lim_{k \to +\infty} \frac{(k+1) \cdot \cancel{(k+1)}}{(2k+3) \cdot 2\cancel{(k+1)}} = \boxed{\frac{1}{4}}$$

As $\rho = \frac{1}{4} < 1$, by the Ratio Test the series is absolutely convergent (AC)

8. (12 pts) Choose ONE to prove:

(a) State and prove the geometric series theorem.

(b) State and prove the p-series test, using the integral test.

See textbook or notes