Name: Solution Key   Panther ID: ____________

Exam 2   Calculus II   Spring 2016

Important Rules:

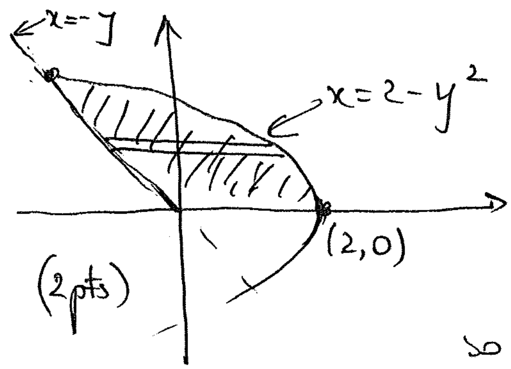
1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (12 pts) Compute the area of the region located above the x-axis, and bounded by $x = 2 - y^2$ and $x = -y$. Sketch and computation are required.



Intersection pts $\begin{cases} x = 2-y^2 \\ x = -y \end{cases} \Rightarrow$

$\Rightarrow -y = 2 - y^2 \Rightarrow y^2 - y - 2 = 0$

$(y-2)(y+1) = 0 \Rightarrow y = 2$ (2pts)

so inters. point in 2nd quadrant ~~$y=-1$~~ ← below x-axis

has coordinates $(-2, 2)$

Best is to divide the region using horizontal stripes (1pt)

$$A = \int_?^? \ell_{stripe} \cdot Th_{stripe}$$

$$Th_{stripe} = dy \quad \cancel{(1pt)}$$

(4pts) $\ell_{stripe} = x_2 - x_1 = 2 - y^2 - (-y) = 2 + y - y^2$

so $A = \int_0^2 (2 + y - y^2)\, dy = \left(2y + \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{y=0}^{y=2} =$ (1pt)

$= 2 \cdot 2 + \frac{2^2}{2} - \frac{2^3}{3} = 6 - \frac{8}{3} = \boxed{\frac{10}{3}}$

(1pt)

2. (32 pts) Compute each of the following:

(a) (10 pts) $\int \sin^4(3x)\cos(3x)\,dx$ $= \int u^4 \frac{1}{3} du = \frac{1}{3} \cdot \frac{u^5}{5} + c$

$u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$\frac{1}{3} du = \cos(3x)\,dx$

$= \boxed{\frac{1}{15} \sin^5(3x) + c}$

(b) (10 pts) $\int_1^e \frac{1}{x(1+(\ln x)^2)}\,dx$ $= \int_0^1 \frac{1}{1+u^2} du = \arctan u \Big|_{u=0}^{u=1}$

$u = \ln x$

$du = \frac{1}{x} dx$

$= \arctan 1 - \arctan 0$

$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$

(c) (12 pts) $\int \arcsin(2x)\, dx$ $=$

IBP $u = \arcsin(2x)$ $dv = 1 \cdot dx$

$du = \frac{2}{\sqrt{1-(2x)^2}}\, dx$ $v = x$

$= x \arcsin(2x) - \underline{\int \frac{2x}{\sqrt{1-4x^2}}\, dx}$

substitution

$w = 1-4x^2$

$dw = -8x\, dx \Rightarrow -2x\, dx = \frac{1}{4} dw$

$= x \arcsin(2x) + \int \frac{\frac{1}{4} dw}{\sqrt{w}}$

$= x \arcsin(2x) + \frac{1}{4} \cdot 2\sqrt{w} + c = \boxed{x \arcsin(2x) + \frac{\sqrt{1-4x^2}}{2} + c}$

3. (28 pts) Set up integrals to represent each of the following (you **do not** have to evaluate).

(a) (8 pts) The arc-length of the curve $y = \ln x$ over the interval $1 \le x \le e$.

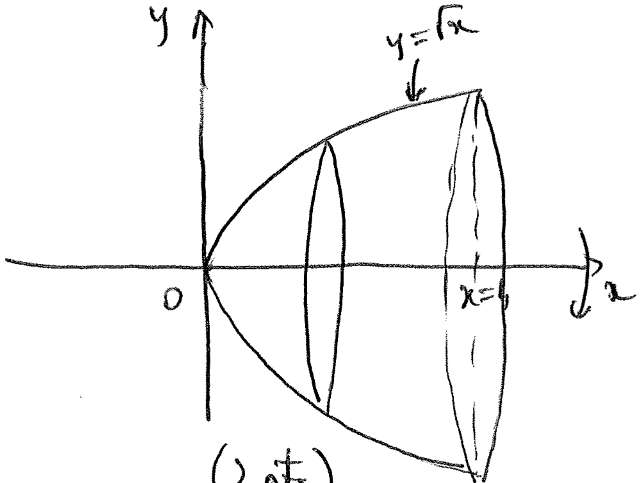
$s = \int_?^? ds = \int_?^? \sqrt{(dx)^2 + (dy)^2} = \int_?^? \sqrt{1 + (f'(x))^2}\, dx$

$y = f(x)$

$dy = f'(x)\, dx$

$s = \int_1^e \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx$

(b) (10 pts) The volume of the solid generated when the region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 4$ is rotated around the x-axis. Sketch of solid is required and specify if you are using slices or cylindrical shells. Just set-up the integral. Computation is **not** required.



(2pts)

Using slices (cyl. shells works as well)

$$V = \int_?^? A_{slice} \cdot Th_{slice} \quad (2pts)$$

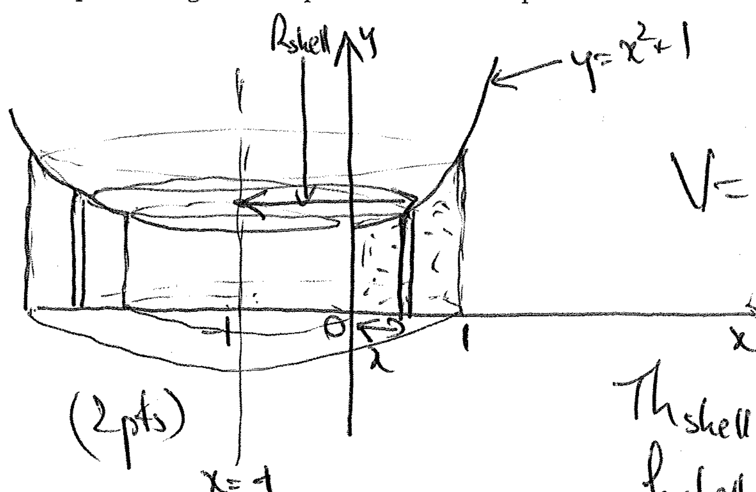
$Th_{slice} = dx$ (1pt)

$A_{slice} = \pi R^2 = \pi \cdot (\sqrt{x})^2 = \pi x$ (1pt)

$R = y = \sqrt{x}$ (2pts)

$$\boxed{V = \int_{x=0}^{x=4} \pi x \, dx = \pi \int_0^4 x \, dx} \quad (2pts)$$

(c) (10 pts) The volume of the solid generated when the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$ is rotated around the line $x = -1$. Sketch of solid is required and specify if you are using slices or cylindrical shells. Just set-up the integral. Computation is **not** required.



(2pts)

Best method: cyl. shells. (1pt)

$$V = \int_?^? 2\pi R_{shell} \cdot h_{shell} \cdot Th_{shell} \quad (2pts)$$

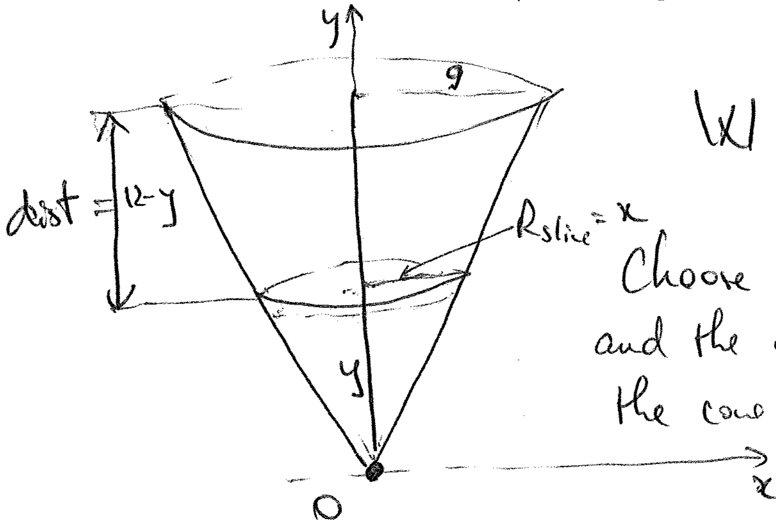
$Th_{shell} = dx$ (1pt)

$h_{shell} = y = x^2 + 1$ (1pt)

$R_{shell} = x - (-1) = x + 1$ (1pt)

$$\boxed{V = 2\pi \int_{x=0}^{x=1} (x+1)(x^2+1) \, dx} \quad (2pts)$$

4. (12 pts) A cone-shaped reservoir has the flat side at ground level and the curved side and the vertex underground. The reservoir is 18 ft in diameter across the top and is 12 ft deep. Assuming that initially the reservoir is full with a liquid of density $\rho = 50$ lbs/ft^3, set up an integral that represents the work necessary to pump out all the liquid through the top of the reservoir. (It is **not** required to evaluate the integral.)



$$W = \int_{?}^{?} dW = \int_{?}^{?} \rho \cdot A_{slice} \cdot Th_{slice} \cdot dist \quad \text{(2pts)}$$

Choose origin at the vertex of the cone and the y-axis be the axis of symmetry of the cone

(2pts)

$Th_{slice} = dy$ (1pt)

$dist = 12 - y$ (2pts)

$A_{slice} = \pi R_{slice}^2 = \pi \cdot x^2$

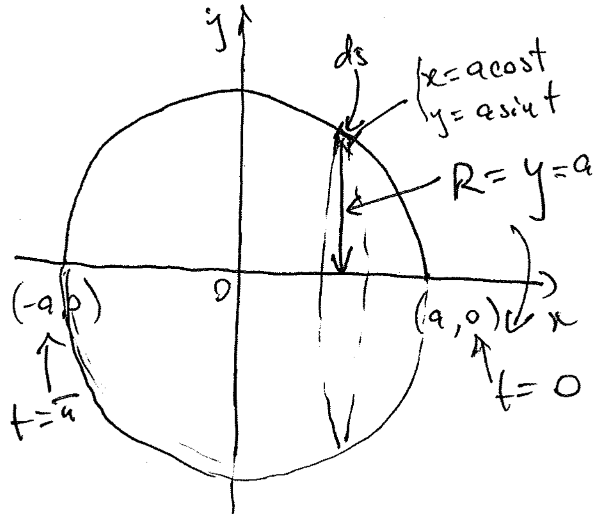
But, by similarity, $\frac{x}{9} = \frac{y}{12} \Rightarrow$

$\Rightarrow x = \frac{9}{12} y = \frac{3}{4} y$,

So $A_{slice} = \pi \cdot \left(\frac{3}{4} y\right)^2 = \frac{9\pi}{16} y^2$ (3pts)

Thus $$\boxed{W = \int_{y=0}^{y=12} 50 \cdot \frac{9\pi}{16} y^2 (12 - y)\, dy}$$ (2pts)

5. (12 pts) Find the formula for the surface area of a sphere of radius a by rotating the semi-circle $x = a\cos t$, $y = a\sin t$, $t \in [0, \pi]$, around the x-axis. Full computation is required.



(1pt)

$$S = \int_{?}^{?} 2\pi R \cdot ds$$ (3pts)

surface area

elem. of arc length

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

R = radius of the piece of curve that is being rotated

since $\begin{cases} x = x(t) \\ y = y(t) \end{cases} \Rightarrow \begin{aligned} dx &= x'(t)\,dt = -a\sin t\,dt \\ dy &= y'(t)\,dt = a\cos t\,dt \end{aligned}$

so $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = a \cdot dt$ (2pts)

$R = y = a\sin t$ (2pts)

Thus

$$S = \int_0^{\pi} 2\pi a \sin t \cdot a\,dt = 2\pi a^2 \int_0^{\pi} \sin t\,dt$$ (2pts)

$$S = 2\pi a^2 \cdot (-\cos t)\Big|_{t=0}^{t=\pi} = 2\pi a^2(-\cos\pi + \cos 0)$$

$$\boxed{S = 4\pi a^2}$$ (2pts)

6. Choose ONE (note the different point values):

(a) (10 pts) State and prove the Integration by Parts formula.

(b) (14 pts) If n is a positive integer, $n \geq 2$, find, with proof, a reduction formula for $\int \sec^n x \, dx$

See notes or textbook