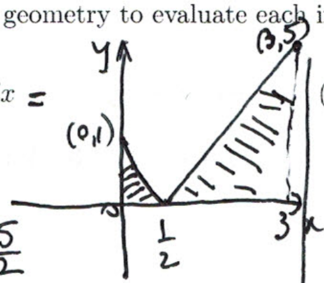
Worksheet week 4 - MAC 2312, Spring 2016 **NAME:** Solution Key

1. Use FTC or geometry to evaluate each integral:

(a) $\int_0^3 |2x-1|\,dx =$



= shaded area

$= \frac{1\cdot\frac{1}{2}}{2} + \frac{5\cdot\frac{5}{2}}{2} =$

$= \frac{1}{4} + \frac{25}{4} = \frac{26}{4} = \boxed{\frac{13}{2}}$

Splitting the integral $\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^3$ and using FTC is also OK, but geometry solution is faster.

(b) $\int_1^2 \frac{x^2+1}{x}\,dx =$

$= \int_1^2 \left(\frac{x^2}{x} + \frac{1}{x}\right) dx =$

$= \int_1^2 \left(x + \frac{1}{x}\right) dx =$

$= \left(\frac{x^2}{2} + \ln x\right)\Big|_{x=1}^{x=2}$

$= (2 + \ln 2) - \left(\frac{1}{2} + \ln 1\right)$

$= \boxed{\frac{3}{2} + \ln 2}$

(c) $\int_0^{\pi/3} \sec^2 x\,dx =$

$= (\tan x)\Big|_{x=0}^{x=\frac{\pi}{3}}$

$= \tan\frac{\pi}{3} - \tan 0$

$= \boxed{\sqrt{3}}$

2. Find the average value of $f(x) = \frac{1}{x^2+1}$ on the interval $[-1,1]$ and find all values of $x^* \in [-1,1]$ so that $f(x^*)$ equals the average value of f on $[-1,1]$. Why is such a value x^* guaranteed to exist?

$f_{ave} = \frac{\int_a^b f(x)\,dx}{b-a}$, so in our case $f_{ave} = \frac{\int_{-1}^1 \frac{1}{1+x^2}\,dx}{1-(-1)} = \frac{(\arctan x)\Big|_{x=-1}^{x=1}}{2}$

$f_{ave} = \frac{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)}{2} = \frac{\frac{\pi}{2}}{2} = \boxed{\frac{\pi}{4}}$

A value x^* in $[-1,1]$ so that $f(x^*) = f_{ave}$ is guaranteed to exist by M.V.T. for integrals, as $f(x) = \frac{1}{x^2+1}$ is continuous on $[-1,1]$.

Solve

$\frac{1}{x^{*2}+1} = \frac{\pi}{4} \iff x^{*2}+1 = \frac{4}{\pi} \iff x^{*2} = \frac{4}{\pi} - 1$, so $\boxed{x^* = \pm\sqrt{\frac{4}{\pi} - 1}}$ are the values in $[-1,1]$ so that $f(x^*) = f_{ave}$

3. Use substitution to compute each integral:

(a) $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{w=1}^{w=2} \frac{1}{\sqrt{w}}\,dw$

$w = \ln x$

$dw = \frac{1}{x}\,dx$

$= \int_1^2 w^{-\frac{1}{2}}\,dw = 2w^{\frac{1}{2}}\Big|_{w=1}^{w=2} =$

$= \boxed{2\sqrt{2} - 2}$

(b) $\int_0^1 \frac{x}{x^2+1}\,dx =$

$u = x^2 + 1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \int_{u=1}^{u=2} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln u\Big|_{u=1}^{u=2}$

$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 = \boxed{\frac{1}{2}\ln 2}$

4. Given that $F(x) = \int_0^x \sqrt{8t - t^2}\, dt$, for $x \in [0, 8]$, do the following:

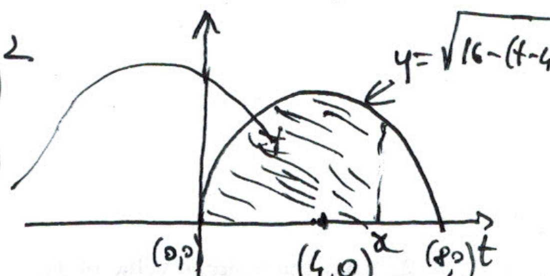
(a) Determine the values of $F(0)$, $F(4)$, $F(8)$. Hint: Complete the square and use geometry.
(b) Determine $F'(x)$ and $F''(x)$.
(c) Based on parts (a) and (b), sketch the graph of the function $y = F(x)$, for $x \in [0, 8]$. What kind of point is $x = 4$ for the graph of $y = F(x)$?

(a) Complete the square for $8t-t^2$

$$8t-t^2 = -(t^2-8t) = -(t^2-2\cdot 4t) = -(t^2-2\cdot 4t+4^2-4^2)$$

$$= -\left[(t-4)^2-16\right] = 16-(t-4)^2$$

Thus $F(x) = \int_0^x \sqrt{16-(t-4)^2}\, dt$: Shaded area



So $F(0) = 0$, $F(4) = \int_0^4 \sqrt{16-(t-4)^2}\, dt = \frac{1}{4}\cdot\pi\cdot 4^2 = 4\pi$

quarter of the area of the disk

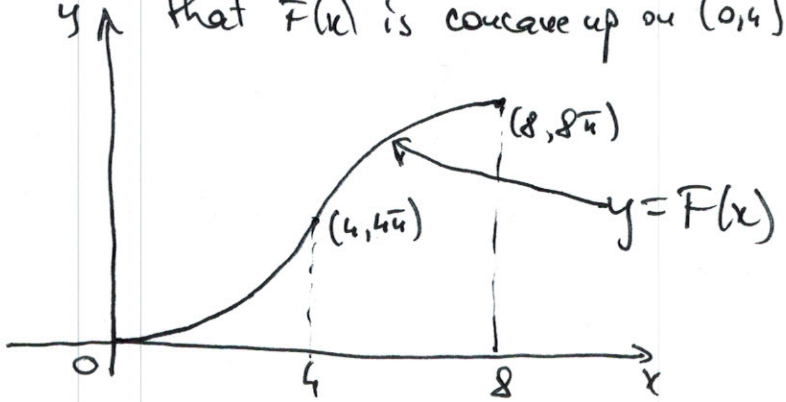
$$F(8) = \int_0^8 \sqrt{16-(t-4)^2}\, dt = \frac{1}{2}\cdot\pi\cdot 4^2 = 8\pi$$

(b) By F.T.C., $F'(x) = \frac{d}{dx}\left(\int_0^x \sqrt{8t-t^2}\, dt\right) = \sqrt{8x-x^2} = (8x-x^2)^{\frac{1}{2}}$

$$F''(x) = (F'(x))' = \left((8x-x^2)^{\frac{1}{2}}\right)' = \frac{1}{2}(8x-x^2)^{-\frac{1}{2}}\cdot(8-2x) = \frac{4-x}{\sqrt{8x-x^2}}$$

(c) Since $F'(x) \geq 0$ for all $x \in [0,8] \Rightarrow F(x)$ is always increasing.

Since $F''(x) > 0$ for $x \in [0,4]$ and $F''(x) < 0$ for $x \in [4,8]$ we know that $F(x)$ is concave up on $[0,4]$ and then concave down on $[4,8]$



Thus $x=4$ is an inflection point

Note also that $F'(0) = F'(8) = 0$

so $x=0$, $x=8$ are critical pts for $F(x)$.