Name: Solution Key Panther ID: ______________

Exam 2 Calculus II Spring 2017

Important Rules:

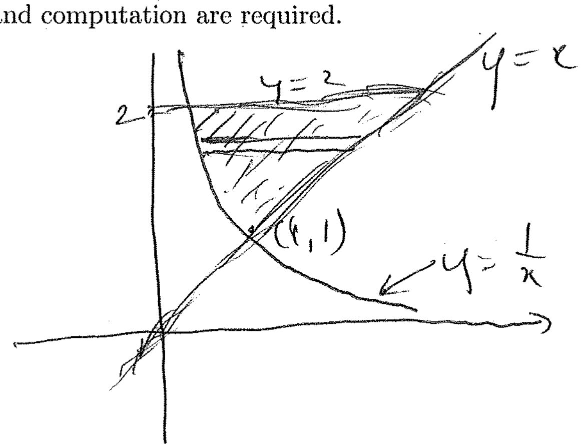
1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) Compute the area of the region in the first quadrant bounded by $y = x$, $y = 1/x$ and $y = 2$. Sketch and computation are required.



Easiest is to use horizontal strips.

$$A = \int_{?}^{?} l_{strip} \cdot th_{strip}$$

$$th_{strip} = dy$$

$$l_{strip} = x_2 - x_1 = y - \frac{1}{y}$$

$$A = \int_1^2 \left(y - \frac{1}{y}\right) dy = \left(\frac{y^2}{2} - \ln y\right)\Big|_{y=1}^{y=2}$$

$$A = \left(\frac{2^2}{2} - \ln 2\right) - \left(\frac{1^2}{2} - \ln 1\right) = \boxed{\frac{3}{2} - \ln 2}$$

2. (20 pts) Compute each of the following:

(a) (10 pts) $\displaystyle\int \frac{\sin(5x)}{3+\cos(5x)}\,dx \quad = \int \frac{-\frac{1}{5}dw}{w} = -\frac{1}{5}\ln|w| + C =$

sub $w = 3+\cos(5x)$

$dw = -5\sin(5x)dx$

$-\frac{1}{5}dw = \sin(5x)dx$

$= -\frac{1}{5}\ln(3+\cos(5x)) + C$

(Note: $3+\cos(5x) > 0$ so the absolute value can be dropped.)

(b) (10 pts) $\displaystyle\int \arctan x\,dx \quad =$

I.B.P. $du = 1\cdot dx$ $\quad v = \arctan x$

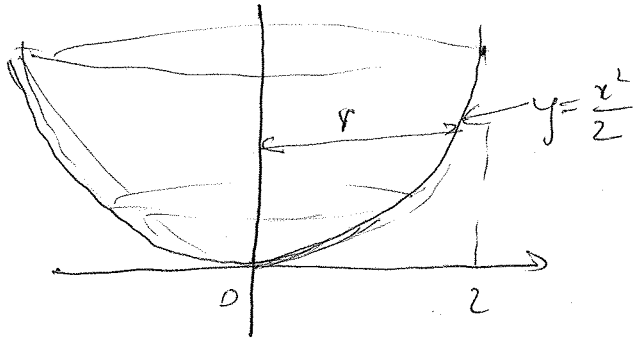
$u = x$ $\quad dv = \frac{1}{1+x^2}dx$

$= x\arctan x - \displaystyle\int \frac{x}{1+x^2}dx =$

either use sub. $w = 1+x^2$ or guess & adjust.

$\boxed{= x\arctan x - \frac{1}{2}\ln(1+x^2) + C}$

3. (14 pts) Find the area of the surface obtained by rotating the curve $y = \frac{x^2}{2}$, with $0 \le x \le 2$, around the y-axis. Full computation is required for this one. (Set-up of the integral - 8 pts, computation of the integral - 6 pts).



$$S = \int_{?}^{?} 2\pi r \cdot \underbrace{ds}_{}$$

element of arc length.

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$r = x \qquad \frac{dy}{dx} = x$$

so $$S = \int_0^2 2\pi x \sqrt{1+x^2}\, dx \underset{\text{subst.}}{=} \pi \int_{u=1}^{u=5} u^{\frac{1}{2}}\, du$$

subst.
$u = 1 + x^2$
$du = 2x\, dx$

$$= \pi \frac{2}{3} u^{\frac{3}{2}} \Big|_{u=1}^{u=5} = \frac{2\pi}{3}\left[5^{\frac{3}{2}} - 1\right]$$

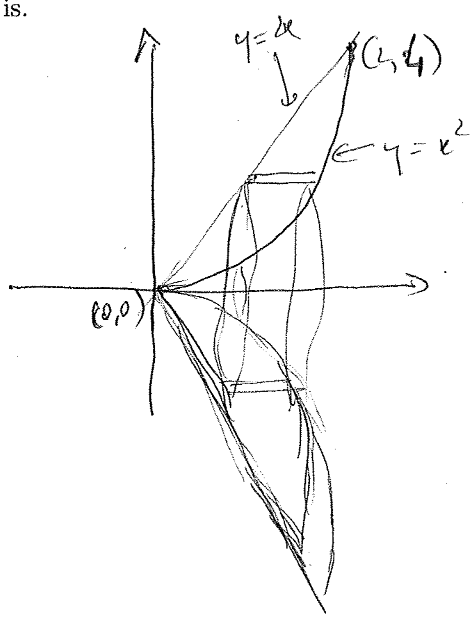
4. (28 pts) Set up integrals to represent each of the following (you **do not** have to evaluate).

(a) (8 pts) The arc-length of the ellipse given in parametric form by $x = a\cos t$, $y = b\sin t$, with $0 \le t \le 2\pi$. Just set-up the integral. Computation is **not** required.

$$L = \int_{?}^{?} ds = \int_{?}^{?} \sqrt{(dx)^2 + (dy)^2} = \int_{?}^{?} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

so $$L = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\; dt$$

(b) (10 pts) The volume of the solid generated when the region bounded by $y = 2x$ and $y = x^2$ is rotated around the x-axis. Specify the method you use. Just set-up the integral. Computation is **not** required, but sketch of the solid is.



① Cylindrical shells

$$V = \int_?^? 2\pi r_{shell} h_{shell} Th_{shell}$$

$$\boxed{V = \int_0^4 2\pi y \cdot \left(\sqrt{y} - \frac{y}{2}\right) dy}$$

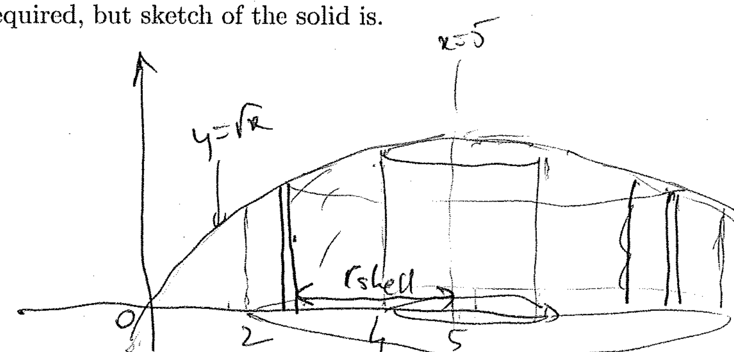
② Slicing (cross section)

$$V = \int_?^? A_{slice} \cdot Th_{slice}$$

$$V = \int_{x=0}^{x=2} \pi\left((2x)^2 - (x^2)^2\right) dx$$

$$\boxed{V = \pi \int_0^2 \left(4x^2 - x^4\right) dx}$$

(c) (10 pts) The volume of the solid generated when the region bounded by $y = \sqrt{x}$ and the x-axis, between $x = 2$ and $x = 4$, is rotated around the line $x = 5$. Specify the method you use. Just set-up the integral. Computation is **not** required, but sketch of the solid is.

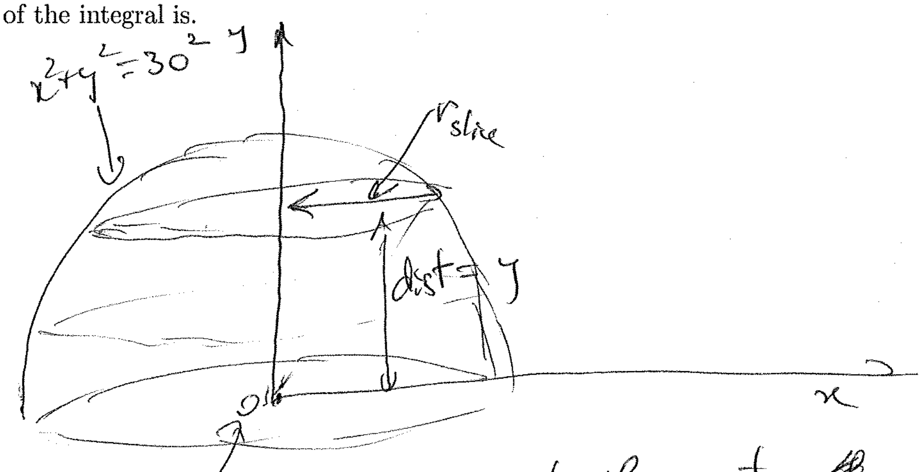


Cylindrical shells is the easier method in this case

$$V = \int_?^? 2\pi r_{shell} \cdot h_{shell} \cdot Th_{shell}$$

$$\boxed{V = \int_{x=2}^{x=4} 2\pi (5-x) \cdot \sqrt{x}\, dx}$$

5. (12 pts) A hemispherical tank has its flat side at ground level and its highest point 30 ft above the ground. (Thus, the base of the tank is a disk of diameter 60 ft). Initially, the tank contains gasoline to a height of 10 ft above the base. Set up an integral that represents the work required to fill up the rest of the tank to its full capacity by a pump at ground level. Assume that gasoline has density $\rho = 45$ lb/ft^3. The calculation is not required, just set-up of the integral is.



Choose origin at the center

$Th_{slice} = dy$

$A_{slice} = \pi r_{slice}^2$

$r_{slice} = x = \sqrt{30^2 - y^2}$

$dist_{slice} = y$

$$W = \int_?^? dW = \int_?^? \rho \cdot A_{slice} \cdot Th_{slice} \cdot dist_{slice}$$

$$\boxed{W = \int_{y=10}^{y=30} 45 \cdot \pi (30^2 - y^2)\, y\, dy}$$

6. (14 pts) (a) (8 pts) Find, with proof, a reduction formula for $\int (\ln x)^n \, dx$, where n is a positive integer.

I.B.P.

$du = 1 \cdot dx \qquad v = (\ln x)^n$

$u = x \qquad dv = n \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx$

$$\boxed{\int (\ln x)^n dx = x \cdot (\ln x)^n - n \int x \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx}$$

So, this is the reduction formula.

(b) (6 pts) Use your reduction formula from part (a) to compute $\int_1^e (\ln x)^2 \, dx$.

$$\int_1^e (\ln x)^2 dx = x \cdot (\ln x)^2 \Big|_{x=1}^{x=e} - 2 \cdot \int_1^e (\ln x)^1 dx =$$

$$= x(\ln x)^2 \Big|_{x=1}^{x=e} - 2 \left[x \cdot \ln x \Big|_{x=1}^{x=e} - 1 \cdot \int_1^e 1 \, dx \right]$$

$$= e (\ln e)^2 - 1 \cdot (\ln 1)^2 - 2 (e \ln e - 1 \cdot \ln 1) + 2(e-1)$$

$$= e \cancel{-2e} \cancel{+2e} - 2 = e - 2$$

7. (12 pts) Choose ONE:

(a) State and prove the Work-Energy theorem.

(b) Using the slicing method, prove the formula for the volume of a pyramid. (if needed, you may assume that the base of the pyramid is a square).

see your notes or textbook