Name: Solution Key Panther ID: ____________

Exam 3 Calculus II Spring 201~~6~~7

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (9 pts) Circle the correct answer. No justification is necessary for this problem (3 pts each).

(a) For the integral $\int \sqrt{9x^2+4}\, dx$, the following substitution is helpful:

(i) $3x = 2\sin\theta$ (ii) $w = 9x^2+4$ (iii) $3x = 2\tan\theta$ (iv) $2x = 3\sec\theta$ (v) $3x = 2\sec\theta$

(Don't spend time evaluating the integral. It is not required.)

(b) Let $f(x)$ be a positive, concave up, continuous function on $[a,b]$ and let T_4 be the trapezoid approximation with 4 subdivisions of the integral $\int_a^b f(x)\, dx$. Then compared with the integral, T_4 is an

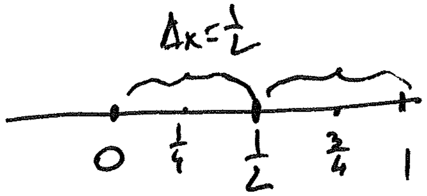
(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about f)

(c) Let $f(x)$ be a positive, concave up, continuous function on $[a,b]$ and let R_4 be the right end point Riemann sum approximation with 4 subdivisions of the integral $\int_a^b f(x)\, dx$. Then compared with the integral, R_4 is an

(i) overestimate (ii) underestimate (iii) exact estimate (iv) cannot tell (more should be known about f)

2. (6 pts) Write an expression representing M_2, the midpoint approximation with 2 subdivisions for the integral below. You do not have to evaluate the expression (nor the integral).

$$\int_0^1 \sin(\pi x^2)\, dx\ .$$



$$M_2 = f\left(\tfrac{1}{4}\right)\cdot\Delta x + f\left(\tfrac{3}{4}\right)\Delta x = \left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right)\right]\cdot\Delta x$$

$$M_2 = \left[\sin\left(\pi\left(\tfrac{1}{4}\right)^2\right) + \sin\left(\pi\cdot\left(\tfrac{3}{4}\right)^2\right)\right]\cdot\frac{1}{2}$$

$$\boxed{M_2 = \frac{1}{2}\left[\sin\left(\frac{\pi}{16}\right) + \sin\left(\frac{9\pi}{16}\right)\right]}$$

3. (4 pts) Write the partial fraction decomposition. It is NOT required to determine the constants.

$$\frac{1}{(x-2)^3(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{x+2} + \frac{Ex+F}{x^2+4}$$

4. (12 pts) In each case, answer True or False and briefly justify (for all, assume $a_k > 0$, for every k)

(a) If $a_k \geq \frac{1}{\sqrt{k}}$ for all $k \geq 1$, then $\sum_{k=1}^{\infty} a_k$ is divergent. **True** **False**

Justification: Simple comparison test and p-series test $\left(\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} \text{ diverges}\right.$ p-series with $\left. p = \frac{1}{2} < 1 \right)$

(b) If $\lim_{k \to +\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

Justification: Counter-example: $\lim_{k\to+\infty} \frac{1}{k} = 0$ but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic series)

(c) If $\lim_{k \to +\infty} \sqrt[k]{a_k} = \frac{2}{3}$ then $\sum_{k=1}^{\infty} a_k$ is convergent. **True** **False**

Justification: By the root test, since

$$\rho = \lim_{k\to+\infty} \sqrt[k]{a_k} = \frac{2}{3} < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ is convergent}$$

5. (10 pts) Evaluate the improper integral or show it diverges $\int_0^{\frac{1}{e}} \frac{1}{x(\ln x)^2}\,dx$

$$\int_0^{\frac{1}{e}} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to0^+} \int_b^{\frac{1}{e}} \frac{1}{x(\ln x)^2}\,dx = *$$

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{w^2}\,dw = -w^{-1} + c = -\frac{1}{\ln x} + c$$

$w = \ln x$

$dw = \frac{1}{x}dx$

$$* = \lim_{b\to0^+}\left(-\frac{1}{\ln x}\Big|_b^{\frac{1}{e}}\right) =$$

$$= \lim_{b\to0^+}\left(-\frac{1}{\ln(\frac{1}{e})} + \frac{1}{\ln b}\right) = 1 + 0 = \boxed{1}$$

Since $\lim_{b\to0^+} \ln b = -\infty$

$\lim_{b\to0^+} \frac{1}{\ln b} = 0$

6. (20 pts) Evaluate each integral

(a) $\int \frac{1}{(4-x^2)^{3/2}}\,dx$ $= \int \frac{dx}{\left(\sqrt{4-x^2}\right)^3} = \int \frac{2\cos\theta\, d\theta}{2^3\cdot\cos^3\theta} =$

$x = 2\sin\theta$

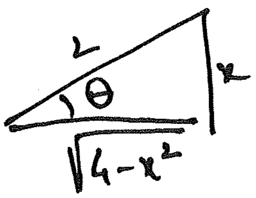
$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$$= \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{4}\int \sec^2\theta\, d\theta$$

$\sin\theta = \frac{x}{2}$

$$= \frac{1}{4}\tan\theta + c$$



$$= \boxed{\frac{1}{4}\frac{x}{\sqrt{4-x^2}} + c}$$

(b) $\int \frac{x^3}{x^2-4}\,dx$ $= \int \frac{x^3 - 4x + 4x}{x^2-4}\,dx = \int\left(\frac{x(x^2-4)}{x^2-4} + \frac{4x}{x^2-4}\right)dx$

$$= \int\left[x + \frac{4x}{(x-2)(x+2)}\right]dx = \frac{x^2}{2} + \int \frac{4x}{(x-2)(x+2)}\,dx$$

could get here also using long division.

Partial fractions for last integral

$$\frac{4x}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

By algebra (or guess & adjust) get $\underline{\underline{A=B=2}}$

so $\int \frac{4x}{(x-2)(x+2)}\,dx = \int\left(\frac{2}{x-2} + \frac{2}{x+2}\right)dx = 2\ln|x-2| + 2\ln|x+2| + c$

so $\int \frac{x^3}{x^2-4}\,dx = \boxed{\frac{x^2}{2} + 2\ln|x-2| + 2\ln|x+2| + c}$

7. (24 pts) Determine if each of the following series converges or diverges. Justify using appropriate tests.

(a) $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ series diverges by the k^{th} term test

$\lim_{k \to +\infty} \frac{k}{2k+1} = \frac{1}{2} \neq 0$ so $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ diverges

(b) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1}$ comparable to $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$

convergent p-series (as $p = \frac{3}{2} > 1$)

As $\frac{\sqrt{k}}{k^2+1} \leq \frac{\sqrt{k}}{k^2}$.

by simple comparison test, $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1}$ is also convergent

(c) $\sum_{k=1}^{\infty} k^3 e^{-k} = \sum_{k=1}^{\infty} \frac{k^3}{e^k}$ Apply Ratio Test

$\rho = \lim_{k \to +\infty} \frac{a_{k+1}}{a_k} = \lim_{k \to +\infty} \frac{(k+1)^3}{\cancel{e^{k+1}}_{e}} \cdot \frac{\cancel{e^k}}{k^3} = \lim_{k \to +\infty} \left(\frac{(k+1)^3}{k^3} \cdot \frac{1}{e} \right) = \frac{1}{e}$

Thus $\rho = \frac{1}{e} < 1$ so

$\sum_{k=1}^{\infty} k^3 e^{-k}$ is convergent by the Ratio Test

8. (15 pts) (a) (6 pts) Sketch the graph of the limacon with inner loop $r = \sqrt{2} + 2\sin\theta$. Be sure to give the polar coordinates of all points where the graph intersects the x-axis, the y-axis, or passes through the origin.

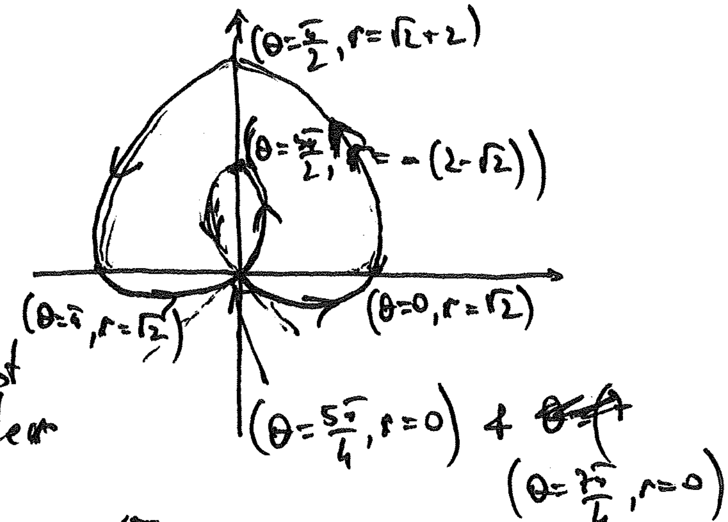
(b) (9 pts) Find the area inside the inner loop of the limacon $r = \sqrt{2} + 2\sin\theta$ (computation required).

(a) Graph will pass through the origin when $r = 0$, i.e.

when $0 = \sqrt{2} + 2\sin\theta \iff \sin\theta = -\frac{\sqrt{2}}{2} \iff \theta = \frac{5\pi}{4},\ \theta = \frac{7\pi}{4}$

θ	r
0	$\sqrt{2}$
$\frac{\pi}{6}$	$\sqrt{2}+1$
$\frac{\pi}{4}$	$\sqrt{2}+2\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}$
$\frac{\pi}{2}$	$\sqrt{2}+2$ ← largest possible
$\vdots$	
π	$\sqrt{2}$
$\frac{5\pi}{4}$	0
$\frac{3\pi}{2}$	$\sqrt{2}-2$
$\frac{7\pi}{4}$	0

negative r when $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$



(b) Using symmetry

$$A = 2\cdot\frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (\sqrt{2}+2\sin\theta)^2\, d\theta$$

$$A = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (\sqrt{2}+2\sin\theta)^2\, d\theta = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (2 + 4\sqrt{2}\sin\theta + 4\sin^2\theta)\, d\theta =$$

$$= \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \left(2 + 4\sqrt{2}\sin\theta + \cancel{4}^{2}\frac{(1-\cos(2\theta))}{\cancel{2}}\right) = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \left[4 + 4\sqrt{2}\sin\theta - 2\cos(2\theta)\right] d\theta$$

$$= \left[4\theta - 4\sqrt{2}\cos\theta - \sin(2\theta)\right]\Big|_{\theta=\frac{5\pi}{4}}^{\theta=\frac{3\pi}{2}} = 4\cdot\left(\frac{3\pi}{2}-\frac{5\pi}{4}\right) - 4\sqrt{2}\left(\cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{5\pi}{4}\right)\right) - \left(\sin(3\pi) - \sin\left(\frac{5\pi}{2}\right)\right)$$

$$= 4\cdot\frac{\pi}{4} - 4\sqrt{2}\cdot\frac{\sqrt{2}}{2} + 1 = \boxed{\pi - 3}$$

9. (10 pts) Choose ONE:

(a) State and prove the p-series test (using the integral test).

(b) Find the positive value of a that satisfies $\displaystyle\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = 1$.

See textbook or notes for part (a)

(b) $$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2\left(\left(\frac{x}{a}\right)^2+1\right)}\,dx = \frac{1}{a^{\cancel{2}}}\cancel{a}\arctan\left(\frac{x}{a}\right)+c$$

so $$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$$

$$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\lim_{x\to+\infty}\arctan\left(\frac{x}{a}\right) - \cancel{\frac{1}{a}\arctan(0)}^{\,0}$$

so $$\int_0^{+\infty} \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{2a}}$$

We want $\frac{\pi}{2a} = 1$ so $\boxed{a = \frac{\pi}{2}}$