Name: Solution Key Panther ID: ____________

Exam 3 MAC-2313 Fall 2018

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (12 pts) Write an appropriate formula for each of the following:

(a) The Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$ of the transformation $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = \underline{\underline{r}}$$

(b) The divergence of a vector field $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$.

$$\operatorname{div}\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

(c) The work done by a force field $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ on a particle that is moving along the curve C, given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, for $t_0 \le t \le t_1$.

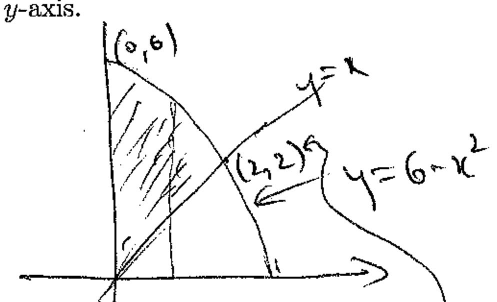
$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C f(x,y)\,dx + g(x,y)\,dy = \int_{t=t_0}^{t=t_1} \left[f(x(t),y(t))\,x'(t) + g(x(t),y(t))\,y'(t)\right]dt$$

(d) A unit normal vector for a parametric surface $\mathbf{r}(u,v)$.

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|}$$

2. (21 pts) Set up iterated double or triple integrals to represent each of the following. Don't spend time trying to evaluate the integrals. It is not required. (Picture is required in each case).

(a) (7 pts) The area of region bounded in the first quadrant by the parabola $y = 6 - x^2$, the line $y = x$ and the y-axis.



$$\boxed{A = \iint_R dA = \int_{x=0}^{x=2} \int_{y=x}^{y=6-x^2} dy\, dx}$$

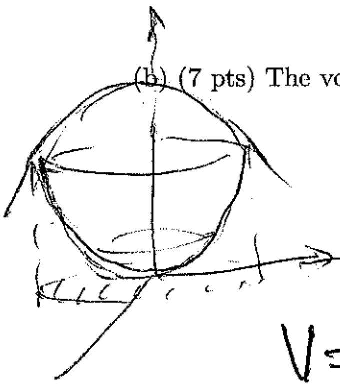
Intersection point

$$\begin{cases} y = x \\ y = 6 - x^2 \end{cases} \Rightarrow 6 - x^2 = x \iff x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0$$

$$\boxed{\begin{cases} x = 2 \\ y = 2 \end{cases}}, \ \cancel{x = -3} \text{ not in first quadrant}$$

(b) (7 pts) The volume of the solid bounded between the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.



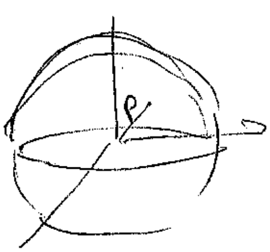
$$\begin{cases} z = 3x^2 + 3y^2 \\ z = 4 - x^2 - y^2 \end{cases} \Rightarrow 4x^2 + 4y^2 = 4 \Rightarrow \begin{cases} x^2 + y^2 = 1 \\ z = 3 \end{cases}$$

← intersection curve of the paraboloids is a circle of radius 1 in the plane $z = 3$

$$\boxed{V = \iiint_G 1\, dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=3r^2}^{z=4-r^2} dz\, r\, dr\, d\theta}$$

↑ use cylindrical coords.

(c) (7 pts) The mass of a spherical solid of radius a if the density is proportional to the distance from the center. (Let k be the constant of proportionality.)



$$m = \iiint_G \delta\, dV \qquad \delta = k \cdot \rho$$

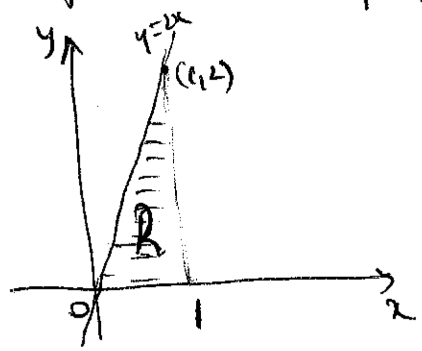
↑ mass ↑ density function

$$\boxed{m = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=a} k \cdot \rho \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta}$$

3. (15 pts) Compute the value of the integral by first reversing the order of integration. Include a picture of the region R.

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$$

Region $R = \{(x,y) \mid \frac{y}{2} \le x \le 1,\ 0 \le y \le 2\}$.



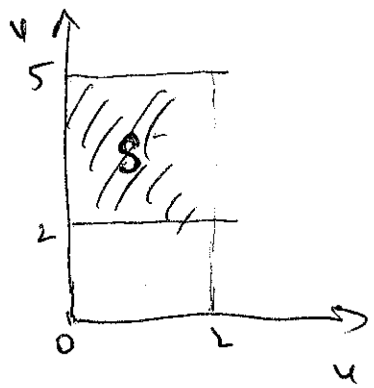
$$\int_{y=0}^{y=2}\int_{x=\frac{y}{2}}^{x=1} e^{x^2}\,dx\,dy = \iint_R e^{x^2}\,dA = \int_{x=0}^{x=1}\int_{y=0}^{y=2x} e^{x^2}\,dy\,dx$$

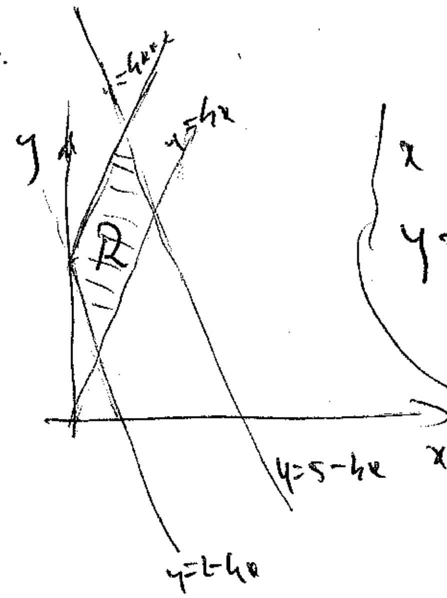
$$= \int_0^1 2x e^{x^2}\,dx = \left(e^{x^2}\right)\Big|_{x=0}^{x=1} = e^1 - e^0 = \boxed{e-1}$$

4. (15 pts) Evaluate the integral

$$\iint_R \frac{y-4x}{y+4x}\,dA, \text{ where } R \text{ is the region enclosed by the lines } \overbrace{y=4x}^{u=0},\ \overbrace{y=4x+2}^{u=2},\ \overbrace{y=2-4x}^{v=2},\ \overbrace{y=5-4x}^{v=5}.$$

Hint: Use the change of variables $u = y - 4x$, $v = y + 4x$.





$$\begin{cases} x = \dfrac{v-u}{8} \\ y = \dfrac{u+v}{2} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{16} - \frac{1}{16} = \boxed{-\frac{1}{8}}$$

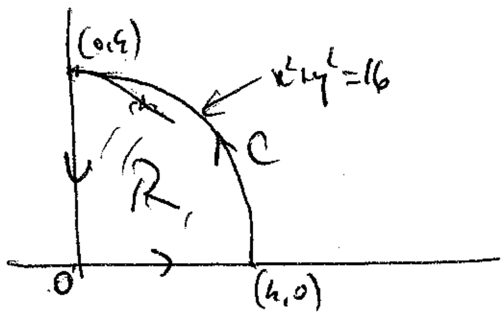
$$\iint_R \frac{y-4x}{y+4x}\,dA_{xy} = \iint_S \frac{u}{v}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv} =$$

$$= \int_{v=2}^{v=5}\int_{u=0}^{u=2} \frac{u}{v}\cdot\frac{1}{8}\,du\,dv = \frac{1}{8}\int_{v=2}^{v=5} \frac{u^2}{2v}\Big|_{u=0}^{u=2} dv =$$

$$= \frac{1}{8}\int_{v=2}^{v=5} \frac{2}{v}\,dv = \boxed{\frac{1}{4}(\ln 5 - \ln 2)}$$

5. (15 pts) Evaluate the line integral $\oint_C x^2y\,dx - y^2x\,dy$, where C is the counter-clock-wise oriented boundary of the region in the first quadrant enclosed by the coordinate axes and the circle $x^2+y^2=16$.

Hint: Easiest is probably to use Green's Theorem, but a direct computation is also possible.



$$\oint_C (x^2y\,dx - y^2x\,dy) \underset{\text{Green}}{=} \iint_R \left(\frac{\partial}{\partial x}(-y^2x) - \frac{\partial}{\partial y}(x^2y)\right) dA_{xy}$$

$$= \iint_R (-x^2-y^2)\,dA = -\iint_R (x^2+y^2)\,dA =$$

$$= -\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=4} r^2\cdot r\,dr\,d\theta = -\int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{r=4} d\theta$$

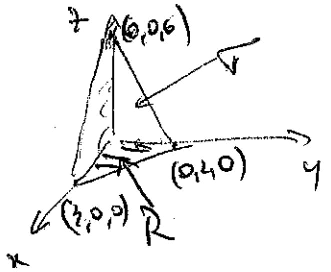
use polar coords

$$= -\frac{4^4}{4}\cdot\frac{\pi}{2} = \boxed{-32\pi}$$

6. (15 pts) Evaluate the surface integral

$$\iint_\sigma (x+y)\, dS\,,$$

where σ is the portion of the plane $z = 6 - 2x - 3y$ in the first octant.



Parametrization for σ:

$\vec{r}(u,v) = (x = u,\ y = v,\ z = 6 - 2u - 3v)$

with $(u,v) \in R$ ← region bounded by the axes of coord. in the first quadrant and the line $6 = 2u + 3v$.

$$\iint_\sigma (x+y)\,dS = \iint_R (u+v) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dA_{uv} = *$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} = +2\vec{i} + 3\vec{j} + \vec{k}$$

$$\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

Thus $$* = \iint_R (u+v)\cdot\sqrt{14}\, dA_{uv} = \sqrt{14} \int_{v=0}^{v=2} \int_{u=0}^{u=3-\frac{3}{2}v} (u+v)\, du\, dv =$$

$$= \sqrt{14} \int_{v=0}^{v=2} \left(\frac{u^2}{2} + uv \right) \Bigg|_{u=0}^{u=3-\frac{3}{2}v} dv = \sqrt{14} \int_{v=0}^{v=2} \frac{\left(3 - \frac{3}{2}v\right)^2}{2} + \left(3 - \frac{3}{2}v\right)v\, dv$$

$$= \sqrt{14} \int_{v=0}^{v=2} \left(\frac{9}{2} - \frac{9}{2}v + \frac{9}{8}v^2 + 3v - \frac{3v^2}{2} \right) dv = \sqrt{14} \int_{v=0}^{v=2} \left(-\frac{3v^2}{8} - \frac{3v}{2} + \frac{9}{2} \right)$$

$$= \sqrt{14} \left(-\frac{v^3}{8} - \frac{3v^2}{4} + \frac{9}{2}v \right) \Bigg|_{v=0}^{v=2} = \sqrt{14} \left(-\frac{2^3}{8} - \frac{3\cdot 2^2}{4} + \frac{9}{2}\cdot 2 \right) = \sqrt{14}(-1-3+9)$$

$$= \boxed{5\sqrt{14}}$$

7. (15 pts) Choose ONE proof. If you do two proofs, only the larger score will be considered for this problem, but the second proof may give some bonus towards a previous problem where your score is smaller. You can use the back of the page.

(A) State and prove the Fundamental Theorem of Line Integrals.

(B) State and prove Green's Theorem for regions with one hole (you can use without proof the Green's Theorem for simply connected regions).

(C) Show that a two-dimensional inverse square field

$$\mathbf{F}(x,y) = \frac{c}{(x^2+y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$$

is conservative in any region in the xy-plane that does not contain the origin and find a potential function $\phi(x,y)$ for $\mathbf{F}(x,y)$.

P. For (A) or (B) see notes or textbook

(C) $\vec{F}(x,y) = c\left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\vec{i} + \frac{y}{(x^2+y^2)^{\frac{3}{2}}}\vec{j}\right)$

Using the test for conservative fields we should check whether $\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{\frac{3}{2}}}\right) \stackrel{?}{=} \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\right)$

But these are both equal to $-\frac{3}{\cancel{2}}(x^2+y^2)^{-\frac{5}{2}}\cdot \cancel{2}xy = -\frac{3xy}{(x^2+y^2)^{\frac{5}{2}}}$

so the vector field is conservative on regions that do not contain $(0,0)$ ($\vec{F}$ is not defined at $(0,0)$)

To find the potential $\Phi(x,y)$, we want

$$\frac{\partial \Phi}{\partial x} = \frac{c\cdot x}{(x^2+y^2)^{\frac{3}{2}}}, \quad \frac{\partial \Phi}{\partial y} = \frac{c\cdot y}{(x^2+y^2)^{\frac{3}{2}}}$$

Integrate the first

$$\Phi(x,y) = \int \frac{cx}{(x^2+y^2)^{\frac{3}{2}}}dx = c(-1)(x^2+y^2)^{-\frac{1}{2}} + g(y)$$

but then $\frac{\partial \Phi}{\partial y} = c(-1)\cdot\left(-\frac{1}{2}\right)(x^2+y^2)^{-\frac{3}{2}}\cdot(2y) + g'(y) = \frac{cy}{(x^2+y^2)^{\frac{3}{2}}}$

so $g'(y) = 0$ so $g(y) = k \leftarrow$ constant

Thus a potential function is

$$\boxed{\Phi(x,y) = \frac{-c}{\sqrt{x^2+y^2}}}$$