Name: Solution key **Panther ID:** ____________

Worksheet 10/11 MAC-2313 Fall 2018

1. The temperature at a point (x, y) on a metal plate in the xy-plane is given by $T(x, y) = 2x^2 - y^3 + x$ degrees Celsius. Assume x, y are measured in centimeters. Suppose a bug is positioned initially at the point $(2, 1)$ on the plate and suppose the bug moves with unit speed.

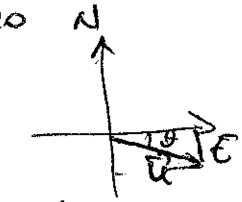
(a) In which direction should the bug go to experience the most rapid increase in temperature? Give your answer as a unit vector, but also as an (approximate) geographical direction.

Should go in the direction of $(\nabla T)_{(2,1)}$

$(\nabla T)(x,y) = (4x+1)\vec{i} - 3y^2\vec{j}$

$(\nabla T)(2,1) = 9\vec{i} - 3\vec{j}$ $\quad \|(\nabla T)(2,1)\| = \sqrt{9^2+3^2} = \sqrt{90} = 3\sqrt{10}$

unit vector $\vec{u} = \dfrac{\nabla T}{\|\nabla T\|} = \dfrac{3}{\sqrt{10}}\vec{i} - \dfrac{1}{\sqrt{10}}\vec{j}$



The bug should go ESE.

More precisely, the bug should go $\approx 18°$ South of East

$\theta = \arctan\left(\frac{1}{3}\right) \approx 18°$

(b) In which direction should the bug go to experience the most rapid decrease in temperature? Give your answer as a unit vector, but also as an (approximate) geographical direction.

The bug should go in the direction of $-\vec{u} = -\dfrac{\nabla T(2,1)}{\|\nabla T(2,1)\|} = -\dfrac{3}{\sqrt{10}}\vec{i} + \dfrac{1}{\sqrt{10}}\vec{j}$

Should go WNW

or $\approx 18°$ North of West

(c) If the bug decides to go straight to the origin (there is a morsel of food there), what rate of change of temperature does it experience just as it starts this trip? What rate of change of temperature does it experience when it is half way there? Does the rate of change of temperature have the same sign during its entire trip? Can you justify your answer?

See the attached page.

(d)* Suppose $(x(t), y(t))$ describe the trajectory of a "heat-seeking" bug moving on the metal plate, where t is the time in seconds. A "heat-seeking" bug always moves in the direction of the most rapid increase of the temperature. Write a system of differential equations that describes the trajectory of a "heat-seeking" bug (for convenience, you can still assume that the bug moves at unit speed, but you can also think what the system would be if you drop this assumption.

We want (or rather the bug wants)

$\vec{r}'(t) = \dfrac{\vec{\nabla} T}{\|\vec{\nabla} T\|}(x(t), y(t))$ for all t $\quad$ (note that $\vec{r}'(t)$ is a unit vector by the assumption that speed $= \|\vec{r}'(t)\| = 1$)

but $\nabla T = (4x+1)\vec{i} - 3y^2\vec{j}$

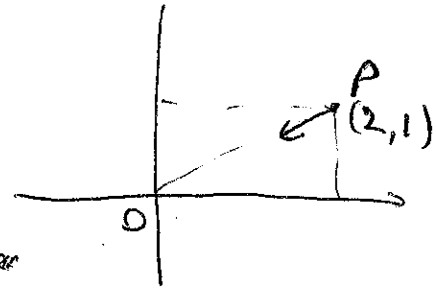
so $\|\nabla T\| = \sqrt{(4x+1)^2 + 9y^4}$

So the system is

$$\begin{cases} x' = \dfrac{4x+1}{\sqrt{(4x+1)^2+9y^4}} \\ y' = -\dfrac{3y^2}{\sqrt{(4x+1)^2+9y^4}} \end{cases}$$

with $x = x(t)$ and $y = y(t)$

Solution for (c)



The bug is moving in the direction of the vector $\vec{PO} = -\vec{OP} = -(2\vec{i}+\vec{j})$.

Since it is moving at unit speed, it is better to parametrize this motion by the unit vector

$$\vec{u} = -\frac{\vec{OP}}{\|\vec{OP}\|} = -\frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}$$

Since the bug starts at $P(2,1)$ the parametric equation of motion are

$$\begin{cases} x(t) = 2 - \frac{2}{\sqrt{5}}t \\ y(t) = 1 - \frac{1}{\sqrt{5}}t \end{cases} \quad \text{with } t \in [0, \sqrt{5}]$$

At the start of the trip, the rate of change of temperature is

$$D_{\vec{u}}T_{(2,1)} = (\nabla T)_{(2,1)} \cdot \vec{u} = (9\vec{i} - 3\vec{j}) \cdot \left(-\frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}\right)$$

$$(D_{\vec{u}}T)_{(2,1)} = -\frac{18}{\sqrt{5}} + \frac{3}{\sqrt{5}} = -\frac{15}{\sqrt{5}} = -3\sqrt{5} \ {}^\circ/\text{cm}$$ ← bug experiences a decrease in temperature

At an arbitrary point on the path

$$(D_{\vec{u}}T)_{(x(t),y(t))} = \left[(4x(t)+1)\vec{i} - 3y(t)^2\vec{j}\right] \cdot \left(-\frac{2}{\sqrt{5}}\vec{i} - \frac{1}{\sqrt{5}}\vec{j}\right)$$

and using $x(t) = 2 - \frac{2}{\sqrt{5}}t$, $y(t) = 1 - \frac{1}{\sqrt{5}}t$ one gets (after computation)

$$(D_{\vec{u}}T)_{(x(t),y(t))} = \frac{1}{\sqrt{5}}\left(-15 + \frac{10t}{\sqrt{5}} + \frac{3}{5}t^2\right)$$

Since $0 \le t \le \sqrt{5}$, along the whole segment $D_{\vec{u}}T \le \frac{1}{\sqrt{5}}(-15+10+3) < 0$,

so, yes, the bug experiences a decrease in temperature during the whole trip, although the rate at which the temperature decrease is smaller and smaller.

Half-way there corresponds to $t = \frac{\sqrt{5}}{2}$ (and point $(1,\frac{1}{2})$)

$$(D_{\vec{u}}T)_{(1,\frac{1}{2})} = \frac{1}{\sqrt{5}}\left(-15 + \frac{10\cdot\sqrt{5}}{2\sqrt{5}} + \frac{3}{5}\cdot\frac{5}{4}\right) = \frac{1}{\sqrt{5}}\left(-10+\frac{3}{4}\right) = -\frac{37}{4\sqrt{5}} \ {}^\circ\text{C/cm}$$

At the origin, corresponds to $t = \sqrt{5}$

$$(D_{\vec{u}}T)_{(0,0)} = -\frac{2}{\sqrt{5}} \ {}^\circ\text{C/cm}.$$