Name: Solution Key [redacted]: ____________________

Exam 1 MAC-2313 Fall 2017

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (20 pts) Given the vectors $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{w} = 2\mathbf{j} - \mathbf{k}$, find each of the following:

(a) the orthogonal projection, $proj_{\mathbf{u}}\mathbf{v}$, of $\mathbf{v}$ on $\mathbf{u}$;

$$proj_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}$$

$$\vec{u}\cdot\vec{v} = -2 + 0 - 1 = -3$$

$$\|\vec{u}\|^2 = 1^2 + (-1)^2 = 2$$

so $\boxed{proj_{\vec{u}}\vec{v} = -\frac{3}{2}(\vec{i} - \vec{k}) = -\frac{3}{2}\vec{i} + \frac{3}{2}\vec{k}}$

(b) the angle between $\mathbf{u}$ and $\mathbf{v}$;

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta \iff \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\|\vec{u}\| = \sqrt{2}$$

$$\|\vec{v}\| = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$\vec{u}\cdot\vec{v} = -3$$

$$\Rightarrow \cos\theta = \frac{-3}{\sqrt{2}\cdot 3} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}}$$

(c) the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$$V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = \left|\begin{vmatrix} 1 & 0 & -1 \\ -2 & 2 & 1 \\ 0 & 2 & -1 \end{vmatrix}\right| =$$

$$= \left| 1\cdot\begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} -2 & 2 \\ 0 & 2 \end{vmatrix} \right| = |-4 + 4| = \boxed{0}$$

This means that the vectors $\vec{u}, \vec{v}, \vec{w}$ lie in the same plane.

(d) a vector $\mathbf{n}$ which is perpendicular to **all** three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Is this possible? Explain.

Because the vectors $\vec{u}, \vec{v}, \vec{w}$ lie in the same plane it is possible to find a vector $\vec{n}$ perpendicular to all of them

Take $\vec{n} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ -2 & 2 & 1 \end{vmatrix} = 2\vec{i} + \vec{j} + 2\vec{k}$

In general, for three vectors in 3-space which do not lie in the same plane (in linear algebra terminology, vectors which are linearly independent) it would not be possible to find a vector perpendicular to all three given vectors.

4. (18 pts) (a) (8 pts) Show that the line L_1 of intersection of the planes $x+2y-z=2$ and $3x+2y+2z=7$ is parallel to the line L_2 given by $x=1-6t$, $y=3+5t$, $z=2+4t$.

~~The~~ A directional vector $\vec{v}_1$ of L_1 is obtained by the cross product of the normals of the given planes (since both normals are perpendicular to $\vec{v}_1$, as $\vec{v}_1$ is in both planes)

$$\vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & 2 & 2 \end{vmatrix} = 6\vec{i} - 5\vec{j} - 4\vec{k}$$

Directional vector for L_2 is $\vec{v}_2 = -6\vec{i} + 5\vec{j} + 4\vec{k}$

As $\vec{v}_2 = -\vec{v}_1$ (scalar multiples), it follows that L_1 and L_2 are parallel.

(b) (10 pts) Find the equation of the plane that contains both lines L_1, L_2 from part (a).

One point P_2 from L_2 (hence from the required plane π) is $P_2(1,3,2)$ (making $t=0$ in param. eqns. of L_2)

To get the normal $\vec{n}$ of our plane, we need two nonparallel vectors in the plane.

One vector is $\vec{v}_2 = \langle -6, 5, 4\rangle$, as is the direction of L_2 (and L_1).

For another vector, we need a point from L_1. We find the parametric eqns. of L_1:

$$\begin{cases} x+2y-z=2 \\ 3x+2y+2z=7 \end{cases} \Rightarrow \begin{cases} 2y-z=2-x \\ 2y+2z=7-3x \end{cases} \Rightarrow \dots \Rightarrow \begin{cases} y=\frac{11}{6}-\frac{5}{6}x \\ z=\frac{5}{3}-\frac{2}{3}x \end{cases},$$

so L_1 in parametric form is $L_1: x=t,\ y=\frac{11}{6}-\frac{5}{6}t,\ z=\frac{5}{3}-\frac{2}{3}t$

For $t=1$, we get the point $P_1(1,1,1)$ on L_1.

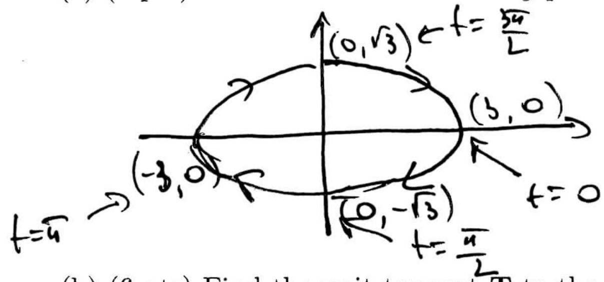
Thus, another vector in the plane is $\vec{P_1P_2} = \langle 0,2,1\rangle$.

$$\vec{n} = \vec{P_1P_2} \times \vec{v}_2 = \dots = 3\vec{i} - 6\vec{j} + 12\vec{k}$$

So the plane is $3(x-1)-6(y-3)+12(z-2)=0$ or... $\boxed{x-2y+4z=3}$

5. (20 pts) Consider the curve in 2-space $\mathbf{r}(t) = 3\cos t\,\mathbf{i} - \sqrt{3}\sin t\,\mathbf{j}$, $t \in [0, 2\pi]$.

(a) (6 pts) Sketch the curve in the xy-plane, indicating the direction of increasing t,



$(x = 3\cos t,\ y = -\sqrt{3}\sin t)$ ellipse

(b) (6 pts) Find the unit tangent $\mathbf{T}$ to the curve at point corresponding to $t = \pi/3$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\vec{r}\,'(t) = -3\sin t\,\vec{i} - \sqrt{3}\cos t\,\vec{j}$$

$$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 3\cos^2 t}$$

$$\vec{T}(t) = \frac{-3\sin t\,\vec{i} - \sqrt{3}\cos t\,\vec{j}}{\sqrt{9\sin^2 t + 3\cos^2 t}}$$

$$\vec{T}\left(\frac{\pi}{3}\right) = \frac{-3\frac{\sqrt{3}}{2}\vec{i} - \frac{\sqrt{3}}{2}\vec{j}}{\sqrt{9\cdot\frac{3}{4} + \frac{3}{4}}} = \frac{\frac{-3\sqrt{3}\,\vec{i} - \sqrt{3}\,\vec{j}}{2}}{\frac{\sqrt{30}}{2}}$$

$$\boxed{\vec{T}\left(\frac{\pi}{3}\right) = \frac{-3\vec{i} - \vec{j}}{\sqrt{10}}}$$

(c) (8 pts) Find the curvature $\kappa(t)$ of the curve.

$$\kappa(t) = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = (x'y'' - y'x'')\,\vec{k}$$

so $\|\vec{r}\,' \times \vec{r}\,''\| = |x'y'' - y'x''|$

$\|\vec{r}\,'\|^3 = \left((x')^2 + (y')^2\right)^{\frac{3}{2}}$

$x' = -3\sin t \quad y' = -\sqrt{3}\cos t$

$x'' = -3\cos t \quad y'' = \sqrt{3}\sin t$

$$\kappa(t) = \frac{|-3\sqrt{3}\sin^2 t - 3\sqrt{3}\cos^2 t|}{(9\sin^2 t + 3\cos^2 t)^{\frac{3}{2}}} = \boxed{\frac{3\sqrt{3}}{(9\sin^2 t + 3\cos^2 t)^{\frac{3}{2}}}}$$

6. (12 pts) Match the following equations with the appropriate surface:

(i) $x^2 + 2y^2 - 3z^2 = 1$ $\longleftrightarrow$ (b)

(ii) $x^2 + 2y^2 - 3z^2 = 0$ $\longleftrightarrow$ (d)

(iii) $(x+1)^2 + 2(y-1)^2 + 3(z-2)^2 = 10$ $\longleftrightarrow$ (a)

(iv) $x + 2y^2 - 3z^2 = 1$ $\longleftrightarrow$ (e)

(v) $2y^2 - 3z^2 = 1$ $\longleftrightarrow$ (f)

(vi) $(x+1)^2 - 2(y-1)^2 - 3(z-2)^2 = 10.$ $\longleftrightarrow$ (c)

(a) ellipsoid (b) hyperboloid with one sheet (c) hyperboloid with two sheets

(d) elliptic cone (e) hyperbolic paraboloid (f) hyperbolic cylinder

7. (10 pts) Find the point(s) of intersection (if any) of the line $x = 1+t, y = 3-t, z = 2t$ with the cylinder $x^2 + y^2 = 16$.

We need to solve the system

$$\begin{cases} x = 1+t \\ y = 3-t \\ z = 2t \\ x^2 + y^2 = 16 \end{cases}$$

$$(1+t)^2 + (3-t)^2 = 16$$

$$1 + 2t + t^2 + 9 - 6t + t^2 = 16$$

$$2t^2 - 4t - 6 = 0 \text{ or } t^2 - 2t - 3 = 0$$

$$\text{or } (t-3)(t+1) = 0$$

So $t=-1$ or $t=3$. Plug each of these in the x, y, z of the line to find two points of intersection

$$\boxed{P_1(0, 4, -2), \quad P_2(4, 0, 6)}$$

8. (12 pts) Choose ONE proof:

(A) Suppose that a particle is moving on a curve with constant speed. Show that at every moment the velocity vector is perpendicular to the acceleration vector.

(B) (Parametric equations of projectile motion) At the initial time $t = 0$ an object is launched from a height y_0 above the ground with an initial velocity vector $\mathbf{v}_0$ which makes an angle α with the horizontal. Starting from Newton's second law, derive the parametric equations of motion. (Assume that the gravitational force is the only force that acts on the object during the entire motion.)

For (B) see notes or textbook

(A) was also done in class, but here is again its solution

(A) Let $\vec{r}(t)$ be the position vector at time t.

Constant speed means $\|\vec{r}\,'(t)\| = \text{const}$.

Thus $\vec{r}\,'(t) \cdot \vec{r}\,'(t) = \|\vec{r}\,'(t)\|^2 = \text{const}$

Take $\frac{d}{dt}$ of both sides of this.

$$\frac{d}{dt}\left(\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right) = \frac{d}{dt}(\text{const}) = 0$$

$$\vec{r}\,''(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$$

$$\cancel{2}\,\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$$

So $\vec{v}(t) \cdot \vec{a}(t) = 0$ so for every t,

$\vec{v}(t) \perp \vec{a}(t)$.