NAME: ____________________________________ **Panther ID:** ______________

Final Exam - MAC 2313 Spring 2010

To receive credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work will not be considered.

1. (30 pts) Set up iterated double or triple integrals to represent each of the following. Don't spend time trying to evaluate the integrals. It is not required. (Picture is required in each case).

(a) (10 pts) The area of the region in the first quadrant inside the circle $x^2 + y^2 = 4$, above the x-axis and below the line $y = x$.

(b) (10 pts) The volume of the solid bounded by the surface $y = x^2$ and the planes $y + z = 4$ and $z = 0$.

(c) (10 pts) Suppose that σ is a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. Show that the surface area of σ is given by

$$\int_\sigma\int dS = \int_R\int \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA\,.$$

Hint: The surface can be parametrized by $x = u, y = v, z = g(u, v)$.

2. (15 pts) The temperature (in degrees Celsius) at a point (x, y) on a metal plate in the xy-plane is $T(x, y) = \dfrac{xy}{1 + x^2 + y^2}$. An ant is located at the point $(1, 1)$ on the metal plate.

(a) (8 pts) Find the rate of change of the temperature at $(1, 1)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$. If the ant decides to move straight to the origin In what (horizontal) direction should you move in order to descend most rapidly? (Answer should be a **unit** vector in the xy plane.)

(b) (7 pts) An ant at (1,1) wants to walk in the direction in which the temperature decreases most rapidly. Find a unit vector in that direction.

3. (20 pts) Write the defining formula for each of the following. Make sure to explain the notations you are using in each case.

(a) Gradient of a function $f(x, y, z)$;

(b) Divergence of a vector field $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + g(x, y, z)\mathbf{k}$;

(c) The speed of a particle whose position vector at time t is $\mathbf{r}(t)$.

(d) The arc length of a curve $\mathbf{r}(t)$, from $t = a$ to $t = b$.

(e) The equation of the tangent plane at a point (x_0, y_0, z_0) of a surface given by $G(x, y, z) = c$, where $G(x, y, z)$ is a function with continuous first-order partial derivatives and c is a constant.

4. (20 pts) Given the points $A(1, 1, 0), B(0, 1, 1), C(1, 0, 1)$ in $\mathbf{R}^3$, find the following:

(a) (6 pts) The angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) (6 pts) The area of the triangle ABC.

(b) (8 pts) The equation of the plane that contains the points A, B, C.

5. (15 pts) Find the absolute maximum and minimum of $f(x, y) = x^2 + 2y^2 - x$ on the closed disk $x^2 + y^2 \leq 1$.

6. (18 pts) Let R be the the region enclosed in the first quadrant by the lines $y = x$, $y = 2x$ and the hyperbolas $xy = 1$ and $xy = 4$. Compute the area of the region R using the change of variables $u = xy$, $v = \frac{y}{x}$.

7. (21 + 7 pts) (a) (14 pts) Show that the 2-dimensional vector field $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - \sin y)\mathbf{j}$ is conservative everywhere and find a potential function.

(b) (7 + 7 pts) Using part (a), or by direct computation, find the work done by $\mathbf{F}(x, y)$ on a particle that moves along the line segment from $(0, 1)$ to $(1, 0)$. (You receive 7 bonus points if you do this part both ways.)

8. (15 pts) Compute the integral

$$\int_C 3xy\;dx + 2x^2\;dy\;,$$

where C is the counter-clock-wise oriented boundary of the region R shown in the picture. R is bounded above by the line $y = x$ and below by the parabola $y = x^2 - 2x$.

9. (15 pts) Find the coordinates of the centroid of the hemisphere $x^2 + y^2 + z^2 = R^2$, $z \geq 0$. Assume that the density is constant, $\delta \equiv 1$. You may use the symmetry of the region to simplify your computations.