Name: ______________________ Panther ID: ____________

Exam 1 - MAC2311 Spring 2014

General Directions: *Read the problems carefully and provide answers exactly to what is requested. Use complete sentences and use notation correctly. Incomprehensible work is worthless. I am grading the work, not just the answer. Don't rush, don't try to do too many steps of a computation at once; work carefully. Good luck!*

1. (10 pts) Consider the function $f(x) = \sqrt{10 - 2x}$.

(a) (3 pts) Find the domain of f. Write your answer in interval form.

$10 - 2x \geq 0 \Rightarrow$

$\Rightarrow -2x \geq -10 \Rightarrow x \leq 5$

Domain of f: $x \in [-\infty, 5]$

(b) (2 pts) Find the range of f. Write your answer in interval form.

$y = \sqrt{10-2x} \geq 0$

Range of f: $y \in [0, +\infty)$

(c) (5 pts) Find a formula for the inverse function $f^{-1}(x)$ and specify its domain.

$y = \sqrt{10-2x}$ Solve for x in terms of y:

$y^2 = 10 - 2x$

$2x = 10 - y^2 \Rightarrow x = 5 - \frac{1}{2}y^2$. Thus $\boxed{f^{-1}(x) = 5 - \frac{1}{2}x^2}$

Domain of f^{-1} = Range of f = $[0, +\infty)$.

2. (10 pts) (a) (5 pts) Solve for x: $\log_{10}(x - 14) - \log_{10} 2 = 3$

$\log_{10}\left(\frac{x-14}{2}\right) = 3 \Rightarrow$

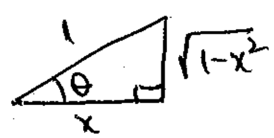
$\Rightarrow 10^{\log_{10}\left(\frac{x-14}{2}\right)} = 10^3 \Rightarrow \frac{x-14}{2} = 1000 \Rightarrow x - 14 = 2000$

$\Rightarrow \boxed{x = 2014}$

(b) (5 pts) Find an equivalent expression, without [redacted] trigonometric functions, for $\tan(\arccos x)$.

$\arccos x = \theta \Leftrightarrow \cos\theta = x = \frac{x}{1}$

$\tan(\arccos x) = ? \Leftrightarrow \tan\theta = ?$



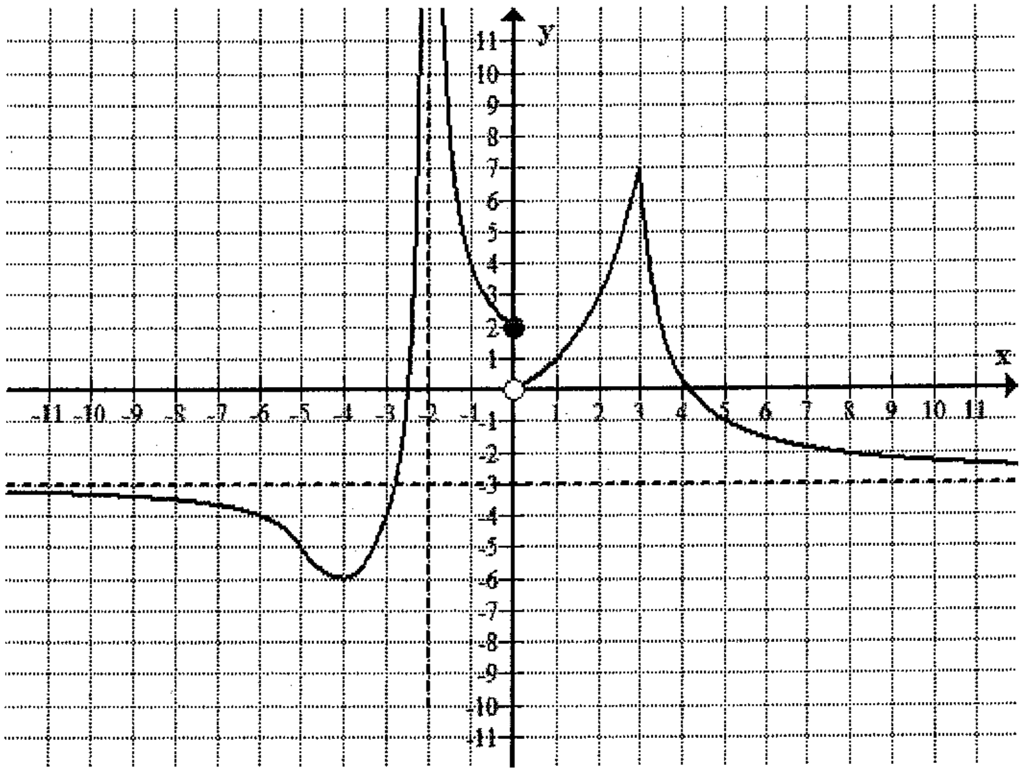
$\Rightarrow \tan\theta = \frac{\sqrt{1-x^2}}{x}$

Thus $\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$

2nd solution!

$\tan(\arccos x) = \frac{\sin(\arccos x)}{\cos(\arccos x)} = \frac{\sqrt{1-\cos^2(\arccos x)}}{\cos(\arccos x)} = \frac{\sqrt{1-x^2}}{x}$

Problem 3. (16 pts) The graph of a function f is given below. Answer the questions that follow.



(8pts) (i) Find the following limits

a) $\lim_{x\to-\infty} f(x) = -3$ b) $\lim_{x\to-2^-} f(x) = +\infty$ c) $\lim_{x\to-2^+} f(x) = +\infty$ d) $\lim_{x\to-2} f(x) = +\infty$

e) $\lim_{x\to 0^-} f(x) = 2$ f) $\lim_{x\to 0^+} f(x) = 0$ g) $\lim_{x\to 0} f(x)$ D.N.E. h) $\lim_{x\to 3} f(x) = 7$

3pts) (ii) Is this function continuous everywhere? If not, at what points (give values of x) is it not continuous? ~~At each point state which condition of continuity fails? (Function has a jump discontinuity is not a reason)~~

Is there any removable discontinuity?

No, f is not continuous at $x=-2$, and at $x=0$

None of them is a removable discontinuity.

3pts) (iii)Is this function differentiable everywhere? If not, at what points (x) it is not differentiable?

No, f is not differentiable at $x=-2$, $x=0$ and at $x=3$.

2pts) (iv) Does f have any asymptote(s)? If yes, what kind? Write their equations.

Yes $x=-2$ is a vertical asymptote

$y=-3$ is a horiz. asymptote

4. (20 pts) Find the following limits. If the limit is infinite or does not exist, specify so. (5 pts each)

(a) $\lim_{x\to 1} \frac{3x-3}{x^2+2x-3} = \frac{0}{0}$

$= \lim_{x\to 1} \frac{3\cancel{(x-1)}}{\cancel{(x-1)}(x+3)} = \boxed{\frac{3}{4}}$

(b) $\lim_{t\to 2^+} \frac{1-2t}{t-2} = \frac{-3}{0^+} = \boxed{-\infty}$

(c) $\lim_{t\to -\infty} \frac{2-t^3}{7t^3+t+3} = \frac{\infty}{\infty}$

$= \lim_{t\to -\infty} \frac{\cancel{t^3}\left(\frac{2}{t^3}-1\right)}{\cancel{t^3}\left(7+\frac{1}{t^2}+\frac{3}{t^3}\right)} = \boxed{-\frac{1}{7}}$

(d) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \frac{0}{0}$ ← multiply up & down by $\sqrt{x}+3$

$= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{\cancel{x-9}\ 1}{\cancel{(x-9)}(\sqrt{x}+3)}$

$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \boxed{\frac{1}{6}}$

5. (14 pts) Compute each of the following limits:

(a) $\lim_{x\to 0} \frac{\sin^2(3x)}{x\tan(2x)} = \frac{0}{0}$

$= \lim_{x\to 0} \frac{\frac{\sin(3x)}{x}\cdot\frac{\sin(3x)}{x}\cdot\cancel{x^2}}{\cancel{x}\cdot\frac{\tan(2x)}{x}\cdot\cancel{x}}$

$= \frac{3\cdot 3}{2} = \boxed{\frac{9}{2}}$

(b) $\lim_{x\to +\infty} \left(x - \frac{x^2}{x+2}\right) = +\infty - \infty$

$= \lim_{x\to +\infty} \left(\frac{x(x+2)}{x+2} - \frac{x^2}{x+2}\right)$

$= \lim_{x\to +\infty} \frac{\cancel{x^2}+2x-\cancel{x^2}}{x+2} = \lim_{x\to +\infty} \frac{2x}{x+2}$

$= \lim_{x\to +\infty} \frac{2\cancel{x}}{\cancel{x}\left(1+\frac{2}{x}\right)} = \boxed{2}$

6. (10 pts) Let $P(t)$ represent the population in millions of a certain country at time t in years where $t = 0$ corresponds to year 2000. For parts (a) and (b), use one sentence to explain in practical terms what each equality is saying.

(a) (2 pts) $P(10) = 19.8$

In 2010, the country has a population of 19.8 million.

(b) (3 pts) $P'(10) = -0.1$

In 2010, the population of the country is decreasing at a rate of 0.1 million per year.

(c) (5 pts) With the information from (a) and (b) estimate the population of this country this year. With one more sentence explain why your estimate may not be entirely accurate.

2014 $\rightsquigarrow t = 14$ so we need to estimate $P(14)$.

Assuming the rate of change of population in (b) be constant

$P(14) \simeq 19.8 - 0.1 \times 4 = 19.4$ million

This may not be accurate, as we assume the rate of change be constant.

7. (10 pts) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \frac{1}{2x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{2(x+h)-1} - \frac{1}{2x-1}}{h} = \lim_{h \to 0} \frac{\frac{2x-1-(2x+2h-1)}{(2(x+h)-1)(2x-1)}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{2x} \cancel{-1} \cancel{-2x} - 2h \cancel{+1}}{(2x+2h-1)(2x-1)} \cdot \frac{1}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{-2\cancel{h}}{(2x+2h-1)(2x-1)} \cdot \frac{1}{\cancel{h}} \right) = \frac{-2}{(2x-1)(2x-1)} = \boxed{-\frac{2}{(2x-1)^2}}$$

8. (10 pts) These are True or False questions. No justification required. No partial credit. 2 points each.

(a) For all $a > 0$, $\sqrt{a^2+4} = a+2$. True **False**

(b) If $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, then f is continuous at $x = 2$. True **False**

(c) The equation $x^3 - 3x + 1 = 0$ has a solution in the interval $[0, 1]$. **True** False

(d) If f is continuous at $x = 2$, then f is differentiable at $x = 2$. True **False**

(e) If $\lim_{x\to 1} f(x) = 3$, then for $x \neq 1$ sufficiently close to 1, $f(x) < 3.1$. **True** False

9. (a) (3 pts) Write the ϵ-δ definition for $\lim_{x\to a} f(x) = L$.

For every $\varepsilon > 0$ there exists $\delta > 0$ so that if $|x-a| < \delta$ and $x \neq a$ then $|f(x) - L| < \varepsilon$

Choose ONE of the parts (b) or (c) and circle the one you try. Only ONE will be graded. Note the different point values.

(b) (7 pts) Use the ϵ-δ definition to show that $\lim_{x\to 2}(5x-7) = 3$.

(c) (12 pts) Use the ϵ-δ definition to show that $\lim_{x\to 2}(5x^2-7) = 13$.

(b) $|f(x) - L| = |5x-7-3| = 5|x-2|$

Given $\varepsilon > 0$, choose $\boxed{\delta = \frac{\varepsilon}{5}}$

Then:

if $|x-2| < \delta \Rightarrow$

$\Rightarrow |f(x) - L| = 5|x-2| < 5\delta = 5\cdot\frac{\varepsilon}{5} = \varepsilon$

q.e.d.

(c) $|f(x) - L| = |5x^2 - 7 - 13| = 5|x^2-4|$

$|f(x) - L| = 5|x-2|\cdot|x+2|$ $(*)$

Preliminary choice: $\delta \leq 1$

if $|x-2| < \delta \leq 1 \Rightarrow -1 < x-2 < 1 \Rightarrow$

$\Rightarrow 3 < x+2 < 5 \Rightarrow |x+2| < 5$

Thus if

$|x-2| < \delta \leq 1 \Rightarrow |f(x) - L| = 5|x-2|\cdot|x+2| < 5\cdot\delta\cdot 5 = 25\delta$ $(**)$

Final choice: $\delta = \min\left(1, \frac{\varepsilon}{25}\right)$ (thus $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{25}$)

With this choice:

if $|x-2| < \delta \Rightarrow |f(x) - L| = 5|x-2|\cdot|x+2| \overset{(**)}{<} 25\delta \leq 25\cdot\frac{\varepsilon}{25} = \varepsilon$

q.e.d.