NAME: Solution Key Panther ID: ____________

Exam 2 - MAC 2311 Spring 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.

2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.

3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations of any type of this rule will lead to a score of 0 on this exam.

4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (10 pts) The function $h(x)$ is given by $h(x) = \frac{x^2}{f(x)}$. It is known that $f(2) = 3$ and $f'(2) = 2$. Compute

(a) (3pts) $h(2)$

$$h(2) = \frac{2^2}{f(2)} = \boxed{\frac{4}{3}}$$

(b) (7pts) $h'(2)$

$$h'(x) = \left(\frac{x^2}{f(x)}\right)' = \frac{2x\,f(x) - x^2\cdot f'(x)}{(f(x))^2}$$

$$h'(2) = \frac{2\cdot 2\cdot f(2) - 2^2\cdot f'(2)}{(f(2))^2}$$

$$h'(2) = \frac{2\cdot 2\cdot 3 - 2^2\cdot 2}{3^2} = \boxed{\frac{4}{9}}$$

2. (10 pts) Show that $y = x\sin x$ satisfies the equation $y'' + y = 2\cos x$.

$$y' = (x\sin x)' = 1\cdot\sin x + x\cdot\cos x = \sin x + x\cos x$$

$$y'' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$$

$$y'' + y = 2\cos x - \cancel{x\sin x} + \cancel{x\sin x} = 2\cos x \checkmark$$

3. (36 pts) Find the derivative of each of the following functions. Simplify your answer when possible (6 pts each):

(a) $y = 5x^4 - \frac{4}{\sqrt{x}} + \frac{e^3}{3}$ $= 5x^4 - 4x^{-\frac{1}{2}} + \underbrace{\frac{e^3}{3}}_{\text{constant}}$

$$y' = 20x^3 - 4 \cdot \tfrac{1}{2} x^{-\frac{3}{2}}$$

$$\boxed{y' = 20x^3 - \frac{2}{\sqrt{x}}}$$

(b) $y = x \arctan(x^2)$

$$y' = (x)' \arctan(x^2) + x\left(\arctan(x^2)\right)'$$

$$y' = 1 \cdot \arctan(x^2) + x \cdot \frac{1}{1+(x^2)^2} \cdot 2x$$

$$\boxed{y' = \arctan(x^2) + \frac{2x^2}{1+x^4}}$$

(c) $y = \ln(\sec x + \tan x)$

$$y' = \frac{1}{\sec x + \tan x} \cdot (\sec x + \tan x)'$$

$$y' = \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x}$$

$$y' = \frac{\sec x\,(\cancel{\tan x + \sec x})}{\cancel{\sec x + \tan x}}$$

$$y' = \sec x$$

So $\left(\ln(\sec x + \tan x)\right)' = \sec x$

(d) $y = \sqrt{1 + \sin^2(3x)}$ $= \left(1 + \sin^2(3x)\right)^{\frac{1}{2}}$

$$y' = \frac{1}{2}\left(1+\sin^2(3x)\right)^{-\frac{1}{2}} \cdot \left(1+\sin^2(3x)\right)'$$

$$y' = \frac{1}{\cancel{2}}\left(1+\sin^2(3x)\right)^{-\frac{1}{2}} \cdot \cancel{2}\sin(3x) \cdot (\sin(3x))'$$

$$y' = \left(1+\sin^2(3x)\right)^{-\frac{1}{2}} \cdot \sin(3x) \cdot \cos(3x) \cdot 3$$

$$y' = \frac{3\sin(3x)\cos(3x)}{\sqrt{1+\sin^2(3x)}}$$

(e) $y = e^{x^e}$

$$y' = \left(e^{x^e}\right)' \overset{\text{chain}}{\underset{\text{rule}}{=}} e^{x^e} \cdot (x^e)'$$

$$y' = e^{x^e} \cdot e \cdot x^{e-1}$$

or

$$y' = e^{(x^e+1)} \cdot x^{e-1}$$

(f) $y = \log_x 2$

With formula of change of base

$$y = \log_x 2 = \frac{\ln 2}{\ln x} = (\ln 2) \cdot (\ln x)^{-1}$$

$$y' = \left((\ln 2) \cdot (\ln x)^{-1}\right)' = (\ln 2) \cdot (-1)(\ln x)^{-2} \cdot \frac{1}{x}$$

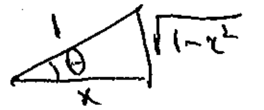
Thus, $y' = -\frac{\ln 2}{x \cdot (\ln x)^2}$

6. (10 pts) Find, with proof, a formula for $(\arccos x)'$.

Let $\theta = \arccos x \Rightarrow \cos\theta = x$ | Differentiate both sides

$$\Rightarrow -\sin\theta \cdot \theta' = 1 \Rightarrow \theta' = -\frac{1}{\sin\theta}$$

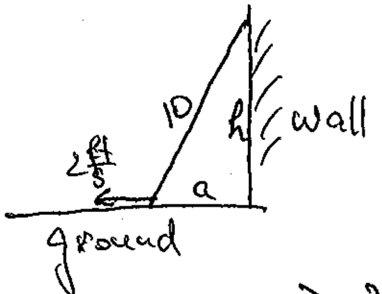
$\cos\theta = x = \frac{x}{1} \Rightarrow$



$\Rightarrow \sin\theta = \frac{\sqrt{1-x^2}}{1}$

$$\Rightarrow \boxed{\theta' = (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}}$$

7. (10 pts) A 10-meter ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 2 m/s, how fast will the top of the ladder be moving down the wall when it is 8 m above the ground?



We know $\frac{da}{dt} = 2\,\frac{m}{s}$. We need to find $\frac{dh}{dt} = ?$

Pythagora $\Rightarrow a^2 + h^2 = 10^2$ | Take $\frac{d}{dt}$

$$\Rightarrow 2a\frac{da}{dt} + 2h\cdot\frac{dh}{dt} = 0 \Rightarrow$$

$$\Rightarrow \frac{dh}{dt} = -\frac{a\frac{da}{dt}}{h}.$$

When $h = 8$, $a = \sqrt{10^2 - 8^2} = 6$

Thus $$\boxed{\left.\frac{dh}{dt}\right|_{\text{when } h=8} = -\frac{6\cdot 2}{8} = -\frac{12}{8} = -\frac{3}{2}\,\frac{m}{s}}$$

8. (10 pts) Find the tangent line at $(2,2)$ to the curve $x^y + y^x = 8$.

Solution without much calculation: Observe the equation is symmetric in (x,y). Thus if a point (a,b) is on the graph, the point (b,a) is also on the graph. This means the graph is symmetric w.r.t. the line $y=x$. At the points of intersection with this line, $(2,2)$ the tangent line must be perpendicular to the line $y=x$.

Hence $m_{\tan \text{ at } (2,2)} = -1$, so $y - 2 = (-1)(x-2)$

$\Rightarrow \boxed{y = -x + 4}$ is the equation of tangent line at $(2,2)$

For solution using implicit diff. see class notes.

5. (14 pts) A certain population of bacteria evolves according to the function $P(t) = 500e^{0.3t}$, where t is the time in hours since an initial moment $t = 0$.

(a) (6 pts) Show that the rate of change of the population is proportional to its size.

$$P'(t) = \left(500e^{0.3t}\right)' = 500 \cdot e^{0.3t} \cdot 0.3 = 0.3\,P(t)$$

Thus the rate of change of population is 0.3 × size of population.

(b) (8 pts) Find the time when the population doubled compared to its initial size and find the rate of change of the population at this time. (Please give units to your answer. OK if the time is left as logarithm.)

$t = ?$ for $P = 2 \cdot P_0$. Note that $P_0 = 500$

so $1000 = 500 \cdot e^{0.3t} \Rightarrow 2 = e^{0.3t} \Rightarrow$

$\Rightarrow \ln 2 = 0.3t \Rightarrow t = \frac{\ln 2}{0.3}$ hours

By (a) $P'\left(\frac{\ln 2}{0.3}\right) = 0.3\,P\left(\frac{\ln 2}{0.3}\right) = 0.3 \times 1000 = 300 \ \frac{\text{bacteria}}{\text{hour}}$

5. (14 pts) (a) (8 pts) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.

(b) (6 pts) At what point(s) in the first quadrant is the tangent line to the Folium of Descartes vertical?

(a) $\frac{d}{dx}(x^3 + y^3) = \underbrace{\frac{d}{dx}(3xy)}_{\text{product rule here}} \Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3y + 3x\frac{dy}{dx} \Rightarrow$

$\Rightarrow 3(y^2 - x)\frac{dy}{dx} = 3(y - x^2) \Rightarrow \boxed{\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}}$

(b) For the tangent line to be vertical, $m_{tan} = \frac{dy}{dx}$ should be undefined. Hence we must have $y^2 - x = 0$. This should be used together with $x^3 + y^3 = 3xy$.

$\begin{cases} x = y^2 \\ x^3 + y^3 = 3xy \end{cases} \Rightarrow (y^2)^3 + y^3 = 3y^2 \cdot y \Rightarrow y^6 - 2y^3 = 0 \Rightarrow$

$\Rightarrow y^3(y^3 - 2) = 0 \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$ origin (not in 1st quadrant)

$\begin{cases} y = \sqrt[3]{2} \\ x = y^2 = \sqrt[3]{4} \end{cases} \Rightarrow \boxed{(x, y) = (\sqrt[3]{4}, \sqrt[3]{2})}$