NAME: Solution Key   Panther ID: ____________

Exam 3 - MAC 2311   Spring 2014

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.
4. Solutions should be concise and clearly written. Incomprehensible work is worthless.

1. (8 pts) For each of the following, fill in the blanks with the most appropriate words or expressions:

(a) If $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is __constant__ on the interval (a, b).

(b) If $f(x)$ is continuous on the interval $[a, b]$ and $f(a) \geq f(x)$ for all $x \in [a, b]$, then the point $x = a$ is an __absolute maximum__ for the function $f(x)$ on the interval $[a, b]$.

(c) A polynomial function of degree 4 can have at most __2__ inflection points.

(d) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a __relative maximum__ for the function.

2. (12 pts) A particle moving along a straight line is accelerating at a constant rate of 5 m/s^2. Find the initial velocity if the particle moves 60 m in the first 4 s.

$a = 5 \frac{m}{s^2}$

$v(t) = at + v_0$

$s(t) = \frac{at^2}{2} + v_0 t + s_0$ ← equations of motion

can take $s_0 = 0$; then $s(4) = 60$

so $60 = \frac{5 \cdot 4^2}{2} + v_0 \cdot 4 \Rightarrow 60 = 40 + 4v_0$

$\Rightarrow \boxed{v_0 = \frac{20}{4} = 5 \frac{m}{s}}$

3. (14 pts) (a) (5 pts) State all indeterminate forms for limits.

$\frac{0}{0}$, $\frac{\infty}{\infty}$ $\quad$ 1^{∞}, 0^0, ∞^0

$0 \cdot \infty$, $\infty - \infty$,

(b) (9 pts) Compute $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$ $\quad$ 1^{∞}

$$\lim_{x \to 0} e^{\ln\left((\cos x)^{\frac{1}{x^2}}\right)} = \lim_{x \to 0} e^{\frac{\ln(\cos x)}{x^2}} = e^{\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}}$$

$$= e^{\lim_{x \to 0} \frac{-\frac{\sin x}{\cos x}}{2x}} = e^{\lim_{x \to 0} -\frac{\tan x}{2x}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

4. (12 pts) Find the absolute maximum/minimum (if they exist) for the function $f(x) = x + \frac{1}{x}$ when $x \in (0, +\infty)$.

$f'(x) = 1 - \frac{1}{x^2}$

Critical pts: $f'(x) = 0 \quad 1 - \frac{1}{x^2} = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1$, $\cancel{x = -1}$ not in the domain

$x = 1$ the only critical pt. in $(0, +\infty)$

$f(1) = 1 + \frac{1}{1} = 2$

$$\left.\begin{array}{l} \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x + \frac{1}{x}\right) = +\infty \\ \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(x + \frac{1}{x}\right) = +\infty \end{array}\right\} \Rightarrow$$

No Abs. max on $(0, +\infty)$

$x = 1$ is the Abs. min for $f(x)$ on $(0, +\infty)$

5. (20 pts) Find each indicated antiderivative :

(a) $\int (5\sin x + \frac{2}{\sqrt{1-x^2}} - \frac{1}{3x})\, dx$

$= -5\cos x + 2\arcsin x - \frac{1}{3}\ln x + c$

(b) $\int x\sec^2(7x^2)\, dx$ $=$

$u = 7x^2$

$du = 14x\,dx$

$\frac{1}{14}du = x\,dx$

$= \int \sec^2(u) \cdot \frac{1}{14}\, du$

$= \frac{1}{14}\tan(u) + c = \boxed{\frac{1}{14}\tan(7x^2) + c}$

(c) $\int x\sqrt{2x-1}\, dx$ $=$

$w = 2x-1 \Rightarrow x = \frac{w+1}{2}$

$dw = 2dx$

$\frac{1}{2}dw = dx$

$= \int \frac{w+1}{2} \cdot \sqrt{w} \cdot \frac{1}{2}dw = \frac{1}{4}\int (w+1)w^{\frac{1}{2}}dw = \frac{1}{4}\int (w^{\frac{3}{2}} + w^{\frac{1}{2}})\, dw$

$= \frac{1}{4}\left(\frac{2}{5}w^{\frac{5}{2}} + \frac{2}{3}w^{\frac{3}{2}}\right) + c = \frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + c$

6. (20 pts) Draw the graph of the function $f(x) = \frac{2x^2+1}{x^2-1}$. Make sure that your work includes these steps.

(a) Determine the domain of the function.

(b) Find eventual vertical asymptotes and determine the behavior of the graph towards the vertical asymptotes (the one-sided limits).

(c) Find eventual horizontal asymptotes.

(d) Find the critical point(s). Using a sign chart for the derivative, determine the intervals over which the function is increasing and the intervals over which is decreasing.

(e) Using the results obtained in parts (a)- (d), draw the graph of the function labeling any eventual asymptotes and the coordinates of the critical point(s).

Note: The analysis of the second derivative and finding inflection points is **not** required. In case you are in doubt about your graph ~~and you need~~ the second derivative to confirm concavity of your graph, here is the second derivative $f''(x) = \frac{6(3x^2+1)}{(x-1)^3(x+1)^3}$. should have been "+"

a) Domain: $x^2 \neq 1$, so $x \neq \pm 1$

all reals except $x=1$, $x=-1$

(b) $\lim_{x\to 1^+} f(x) = \lim_{x \to 1^+} \frac{2x^2+1}{x^2-1} = \frac{3}{0^+} = +\infty$; $\lim_{x\to 1^-} \frac{2x^2+1}{x^2-1} = \frac{3}{0^-} = -\infty$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{2x^2+1}{x^2-1} = \frac{3}{0^-} = -\infty$; $\lim_{x\to -1^-} \frac{2x^2+1}{x^2-1} = \frac{3}{0^+} = +\infty$

$x=1$, $x=-1$ are V.A.

(c) $\lim_{x\to \pm\infty} \frac{2x^2+1}{x^2-1} = 2 \Rightarrow y=2$ is horiz. asymptote (H.A.)

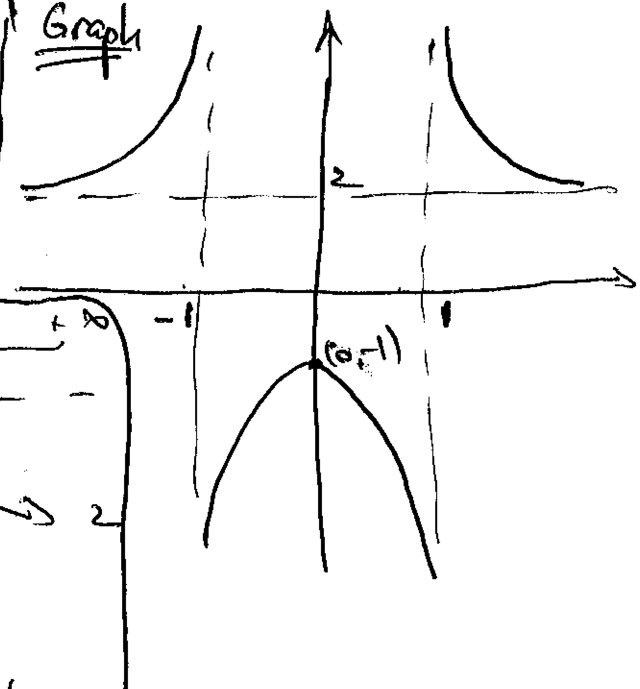
(d) $f'(x) = \frac{4x(x^2-1)-(2x^2+1)\cdot 2x}{(x^2-1)^2} = \frac{-6x}{(x^2-1)^2}$

$x=0$ critical point

rel. max.

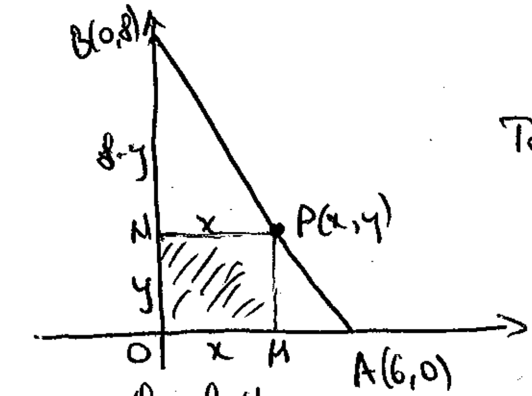
x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		+ + + + +	‖	+ + + +	0	− − −	‖	− − − −	
$f(x)$	2	↗	$+\infty$ ‖ $-\infty$	↗	-1	↘	$-\infty$ ‖ $+\infty$	↘	2

Graph



It also helps to notice that the function is even, hence the graph is symmetric about the y-axis

7. (12 pts) In the picture, point A has coordinates $(6,0)$, point B has coordinates $(0,8)$. A variable point P on the segment AB is projected on the coordinate axes to determine (variable) points M and N. Find the coordinates of the point P so that the rectangle $OMPN$ has the largest possible area.



$A = x \cdot y$

To find the relation between (x,y) you can either write the equation of line AB (you know two points) or use similarity.

I use the latter:

$\Delta BNP \sim \Delta BOA$

$\Downarrow$

$\frac{8-y}{8} = \frac{x}{6} \Rightarrow 8-y = \frac{8}{6}x$

$\Rightarrow y = 8 - \frac{4}{3}x$

$\rightarrow A(x) = x \cdot (8 - \frac{4}{3}x)$ Maximize when $x \in [0,6]$

$A(x) = 8x - \frac{4}{3}x^2$

$A'(x) = 8 - \frac{8}{3}x$

$A'(x) = 0 \Rightarrow 8 = \frac{8}{3}x \Rightarrow x = 3$

$x=3$ is the abs. max, since $A(0) = 0$, $A(6) = 0$

So, the ideal point P has coordinates $\boxed{x=3, y=4}$.

8. (12 pts) (a) (6 pts) State the Mean Value Theorem.

If $f(x)$ continuous on $[a,b]$ and differentiable on (a,b) there is a point $c \in (a,b)$ so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(b) (6 pts) Use the Mean Value Theorem to show that

$$|\sin a - \sin b| \le |a - b|\,, \text{ for any real values } a, b.$$

If $a=b$, both sides are equal to 0, so the statement is true

If $a \neq b$ we apply MVT for $f(x) = \sin x$ on the interval $[a,b]$ (assume $a<b$)

Then $\frac{\sin b - \sin a}{b-a} = \cos c$ for some $c \in (a,b)$

Thus $\frac{|\sin b - \sin a|}{|b-a|} = |\cos c| \le 1$ as $\cos c \in [-1,1]$

Thus $|\sin b - \sin a| \le |b-a|$